Tarski-Banach and the Atomic Nature of Matter

The Tarski-Banach Theorem says that a sphere can be decomposed into a finite number of pieces which can then be recombined to form two spheres, each identical to the original sphere. They gave an explicit construction using just 5 pieces. It is important to realise that the two spheres thus formed really are identical to the original. They are the same size and they have no ‘seams’ or cavities.

A second Tarski-Banach Theorem holds that any sphere can be broken down into a finite number of pieces and re-assembled into a sphere of different size – any size.

On first hearing these theorems they seem completely bonkers. On examination, the reason for this revolt of our intuition lies in physics rather than maths. The theorems seem to violate the conservation of mass: you can’t get something for nothing. But on reflection they are less mysterious.

One of the precursor results which was established (by Hausdorff) is that the unit interval can be decomposed into a countable number of pieces and re-assembled into the interval of length 2. The uncountable infinity of the mathematical continuum is remarkably elastic. Tarski-Banach is similar in nature, but the fact that it is 3-dimensional invites interpretation in the physical world. It is at this point that the weirdness sets in.

The real lesson that Tarski-Banach teaches physicists, in my view, is that matter could not possibly have been continuous. I never quite followed the ancients’ arguments for the atomic nature of matter, but their intuition was good in this instance. Tarski-Banach seems to tell us that, had matter been a mathematical continuum, then the conservation of mass could not apply. We would end up being able to get something for nothing. So, matter must be atomic.

Another way of expressing this is as follows. If mass is to be conserved then it must first be countable, since otherwise what is it that is conserved? Being countable it is therefore atomic, i.e. not a continuum.