T73S02 Session 22 – Fatigue Crack Growth

Last Update 22/1/16

Fatigue crack growth: derivation of the Paris Law; Derivation of the fatigue threshold stress intensity factor range; Partial closure and the effective SIF range

Qu.: What is fatigue crack growth?

Cyclic loading can cause a crack to extend a little on each cycle, even though the maximum K or J is far smaller than the initiation toughness. This is fatigue crack growth (fcg).

Qu.: If K is less than the initiation toughness, how can fcg happen?

Remember that any fracture process is ultimately about energy being supplied to the crack tip region to drive some damage mechanism. Fatigue crack growth is driven by the energy absorbed by the reversing plastic zone around the crack tip, i.e., the hysteresis cycle. Every load cycle supplies more energy into this region to drive the crack growth by another increment.

Qu.: Is all fcg described by the same empirical law?

No, there are a few different regimes. Broadly these are,

- Small cracks;
- LEFM or small scale yielding (SSY) regimes;
- Widespread yielding (WSY) regime.

Qu.: What is the ‘small crack’ fatigue regime?

The reversing plastic zone which is responsible for driving the fcg might be present even in the absence of the crack. In other words, the uncracked body stresses might exceed twice yield in some localised region such as near a notch, or at the sharp corner of a weld toe. The essence of the ‘small crack’ regime of fcg is that it is driven by the pre-existing cyclic plastic zone. The same cyclic plastic zone might be responsible for the crack forming in the first place (R5V2/3).

The small crack fcg law in R66 is,

$$\frac{da}{dN} = B'a^{Q} \text{ mm/cycle}$$

where a is the crack depth in mm. R66 provides recommendations for the parameters B' and Q for a range of materials. An upper bound for all these materials is given by $Q = 1$ and $B' = 26{,}100\Delta\varepsilon_t^{2.85}$ for a total strain range of $\Delta\varepsilon_t$ (absolute). This bounds data obtained at high temperatures (550^{o}C and above).

We will not say more about this small-crack regime here – see R66.

Qu.: What is the SSY regime of fcg?

In the SSY regime, the reversing, cyclic, plastic zone is due to the crack tip field itself – and the crack tip fields are those relating to small scale yielding. This means that there is a region surrounding the crack tip plastic zone in which the stress and strain fields approximate to LEFM form. So the growth laws applicable in the SSY regime also apply in the LEFM regime.

In this regime the crack will grow by fatigue under the influence of its self-generated, crack-tip, cyclic plasticity. It is reasonable to expect the fatigue crack growth per cycle to be related to the stress intensity factor range, ΔK.

Qu.: What is the empirical form of fcg law in the SSY regime?

Empirically, structural metals are found to have fatigue crack growth rates which vary with load and crack size according to $\frac{da}{dN} \propto \Delta K^m$, where the exponent, *m*, generally lies between 3 and 4. This is the ***Paris Law***.

Qu.: Why does fcg obey a Paris Law?

Qu.: Why does *m* take a value of about 3 or 4?

Why is *m* not 1 or 10, for example? It is instructive to consider some simple models which demonstrate where a Paris-type law comes from, and also why the exponent might be expected to be in the range 3 to 4. The models are restricted to the small scale yielding regime, where the plastic zone around the crack tip is small compared with the remaining ligament. We also assume for simplicity that all the cycles are of the same magnitude.

On first loading, a plastic zone is formed around the crack tip. When the load is removed, most of the material within the crack tip plastic zone simply unloads down the elastic line. In other words, most of the initial plastic zone cycles elastically after first yielding. This is because most of the material within the initial plastic zone will be at less than twice the yield stress (when calculated elastically). Roughly speaking, it is only the material whose elastic stress exceeds twice the yield stress which will fall within the zone of reversing, cyclic plasticity. The cyclic plastic zone is much smaller than the initial plastic zone. Only the material within the cyclic plastic zone absorbs energy repeatedly on every cycle and hence drives fatigue crack growth.

The crudest model of material behaviour is this: assume that within the cyclic plastic zone the stress and the strain both take some uniform "yield" values, σ_0 and ε_0, and, on unloading, take equal and opposite values[1]. Further assume that the size of the cyclic plastic zone can be estimated from the LEFM fields by requiring $\Delta\sigma = 2\sigma_0 = \frac{\Delta K}{\sqrt{2\pi r_Z}}$, so that the cyclic plastic zone has size,

$$r_Z = \frac{\Delta K^2}{8\pi\sigma_0^2} \qquad (1)$$

[We have used the Mode I stress for convenience. Using the Mises equivalent just produces a slightly different numerical constant in (1), and will depend upon plane stress or plane strain].

The energy deposited per unit volume is just $2\sigma_0\varepsilon_0$. Crudely assuming that the plastic zone is a circle centred on the crack tip, the energy deposited per cycle is thus,

$$\text{energy deposited per cycle} = \frac{\varepsilon_0 t \Delta K^4}{32\pi\sigma_0^3} \qquad (2)$$

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[1] These can be thought of as "signed equivalent" quantities, following R5.

where t is the thickness (i.e. the length of crack front considered). Assuming it requires a fixed amount of energy to grow the crack by a unit increment of area (G_c), the crack growth per cycle is therefore,

$$\frac{da}{dN} = \frac{\varepsilon_0 \Delta K^4}{32\pi\sigma_0^3 G_c} \tag{3}$$

This is a fatigue crack growth law of Paris type.

The exact form of Equ.(3) should not be used in assessments! The parameters *m* and *C* in the Paris Law, $\frac{da}{dN} = C\Delta K^m$, should be found from fatigue crack growth tests on the material of interest. The only purpose of the above derivation of (3) is to rationalise the typical magnitude of the exponent, *m*, in this case m = 4.

The area of the plastic zone which led to (2), and hence (3), was very crude. It will not really be a circle centred on the crack tip. But the only difference that will occur if the correct area is used is that the numerical constant in (3) will be different. It will not affect the conclusion that the exponent, *m*, is 4.

Qu.: How well does (3) agree with empirical values of *C* and *m*?

R66 gives the following parameters for upper bound fatigue growth laws of Paris Law type (units: ΔK in MPa√m gives crack growth in m/cycle), are:-

Material	**C** (20°C)	**m**
CMn	1.5×10^{-11}	3
CMn (ST)	4×10^{-12}	4
CMV	4.1×10^{-11}	3.5
316 parent	1.4×10^{-10}	3
316 weld	9.1×10^{-10}	3

So, our crude model giving $m \sim 4$ is not bad, though a value of $m \sim 3$ appears more typical.

For fun, we can ask if the magnitude of the coefficient in Equ.(3) also aligns roughly with R66 data. For this purpose it is necessary to use an empirical law which has $m = 4$, otherwise the comparison of the coefficient in (3) would be meaningless. This applies for CMn steels in the short-transverse orientation.

For illustration assume a typical CMn yield stress of 250 MPa at room temperature and an elastic strain of 0.125%, and hence E = 200 GPa. Assume a best-estimate fracture toughness of 160 MPa√m, which implies $G_c \sim 0.13$ MPa.m. The coefficient in (3) thus evaluates to $\sim 5 \times 10^{-12}$ for growth in metres and ΔK in MPa√m. This is in remarkably good agreement with the value in R66 at ambient temperature (4×10^{-12}). Probably this is a lucky fluke, but it supports the credibility of the mechanism.

Qu.: Is the model improved by using the HRR fields instead of perfect plasticity?

One of the especially naïve features of the above derivation (though there are many) is the assumption of a uniform "yield" stress and strain within the cyclic plastic zone. A more realistic approach is to assume the HRR fields apply. Recalling that J/σ has units of length, the stress and strain fields vary with distance, r, from the crack tip as,

$$\sigma \propto \sigma_0 \left(\frac{J}{\sigma_0 r}\right)^{\frac{1}{n+1}} \quad \text{and} \quad \varepsilon \propto \varepsilon_0 \left(\frac{J}{\sigma_0 r}\right)^{\frac{n}{n+1}} \tag{4}$$

where power law hardening is assumed, $\frac{\varepsilon}{\varepsilon_0} = \left(\frac{\sigma}{\sigma_0}\right)^n$. Instead of a uniform energy deposition per unit volume we now have a deposition which is more intense near the crack tip,

$$\text{energy deposited per unit volume} \propto \sigma\varepsilon = \frac{\varepsilon_0 J}{\sigma_0 r} \propto \frac{\Delta K^2}{r} \tag{5}$$

To find the total energy deposited we therefore need to integrate (5) over the volume of the cyclic plastic zone. The volume element is $2\pi rt.dr$. Hence we get,

$$\text{energy deposited per cycle} \propto tr_z \Delta K^2 \tag{6}$$

where r_z is the cyclic plastic zone size, again assumed circular and centred on the crack tip. This is found by equating the PYFM stress to twice yield, i.e.,

$$\sigma = 2\sigma_0 \propto \sigma_0 \left(\frac{J}{\sigma_0 r}\right)^{\frac{1}{n+1}} \tag{7}$$

but this once again gives $r_z \propto \Delta J \propto \Delta K^2$, as in the LEFM case, Equ(1). Substituting this into Equ.(6) gives,

$$\text{energy deposited per cycle} \propto \Delta K^4 \tag{8}$$

So the exponent m = 4 in the Paris law persists when PYFM crack tip fields are assumed.

Qu.: Why, for most structural steels of interest, do fatigue tests tend to give *m* closer to 3 than 4?

A naïve assumption which has persisted in the last model is that the cyclic plastic zone is the region over which the initial stresses exceed twice yield, where ‘yield’ is taken as the initial yield stress σ_0. But this takes no account of hardening, or the possible distinction between isotropic hardening and kinematic hardening, or other variants. Actually, a little thought reveals that taking account of the effects of work hardening on the ‘yield’ strength, whatever the hardening law used for reversing plasticity, makes no difference to the conclusion: *m* remains stubbornly at 4.

The same applies if one examines the effect of the shape of the twice-yield region – which will actually differ greatly from a circle (and is more dumb-bell shaped for plain strain) - *as long as we assume contained yielding*. Even quite general formulations of the above model seem always to arrive at an expression for da/dN which is proportional to $\Delta J^2 \propto \Delta K^4$.

So why is *m* generally closer to 3? I suspect the reason is that the small scale yielding assumption is not quite accurate for the tests typically carried out on structural steels. In the above derivations the growth rate results from two factors: (i) the energy deposited per cycle per unit volume, which is proportional to ΔK^2, and, (ii) the volume of the cyclic plastic zone. In the small scale yielding regime the latter is also proportional to ΔK^2, so the overall growth rate is predicted to be proportional to ΔK^4.

However, if the plastic zone is large enough to be affected by the gross plastic mechanism (think slip line fields), the shape of the plastic zone may no longer scale in this way. It may, for example, get longer if ΔK were increased, but not much wider. Alternatively, if the yielding becomes general. the plastic zone cannot get any longer, but may get wider as ΔK is increased further. In both these cases the volume of the plastic zone, and by analogy the cyclic plastic zone, is less sensitive to ΔK than previously - so it is reasonable that the previous ΔK^2 dependence might become more like a ΔK dependence. So, overall, the growth rate then goes as more like ΔK^3. If this explanation is correct, a ΔK^4 would prevail for modest ΔK levels. I don't know if this is so.

Qu.: What about the widespread yielding (WSY) regime?

We will not address this in detail here – see R5 and R66. However, the main difference between SSY and WSY is that it is no longer appropriate to use fcg laws expressed in terms of ΔK. Physically this is because ΔK fails to capture the influence of the widespread yielding on the cyclic plastic strains, and hence on the fcg rate. The relevant parameter for WSY is ΔJ, which takes the elevated reference strain range into account.

The fcg law may be essentially the same as for SSY, but re-expressed in terms of ΔJ, i.e., $\frac{da}{dN} = C(E'\Delta J)^{m/2}$. The cyclic J-integral, ΔJ, can be found by replacing stress, strain and displacement in the usual J-integral by their ranges. Alternatively it can be found from the reference stress formula, $\Delta J = \frac{\Delta\varepsilon_{ref}}{\Delta\sigma_{ref}}\Delta K^2$ .See R5V4/5, Appendix A3 for details.

Qu.: How is ΔK for mixed modes defined when using the Paris Law?

Currently I don't know what R5/R66 intends to imply for mixed mode situations. Provisionally I suggest using the effective SIF range $\sqrt{\Delta K_I^2 + \Delta K_{II}^2 + \Delta K_{III}^2/(1-\nu)}$ but this is subject to an R5 Query. Check User Queries & latest R6/R66 - any update?

Qu.: Partial closure and the effective SIF range

When is a crack not a crack? When it is closed. If the stresses were always compressive over the crack then the crack might as well not be there – and in particular fatigue crack growth would not be expected. More commonly the applied stress may range between compressive and tensile values. So the crack may be closed over part of the load cycle and open over the rest of the cycle. How does this affect fcg? The advice in R5 and R66 is as follows: define the minimum and maximum SIFs as $K_{\min}$ and $K_{\max}$, where the latter would be negative for compressive loading. Put $R = K_{\min}/K_{\max}$ and then use $\Delta K_{eff} = q\Delta K$ in the Paris law to evaluate fcg, where,

$$q = 1 \qquad \text{for } R \ge 0 \tag{9a}$$

$$q = \frac{1 - 0.5R}{1 - R} \qquad \text{for } R < 0 \tag{9b}$$

Qu: Does this apply if the crack has a wide (unloaded) gape?

No.

Note that there is an assumption implicit in the advice to use $\Delta K_{eff} = q\Delta K$, namely that the crack was not opened before being loaded. If the crack has a substantial gape under zero load, then it may be that the *whole* SIF range, ΔK, contributes to fcg because the crack faces are never in contact, even under the greatest compressive load. In this case you should use $\Delta K_{eff} = \Delta K$.

Qu.: What effect does environment have on fcg rates?

R66 data was obtained predominantly from tests in air, with some tests also in vacuum or steam.

Until recently we have generally assumed in assessments that a CO_2 atmosphere is benign. Unfortunately, we have realised recently that there are data for fcg rates in CO_2 which indicate that rates are faster than in air. This is a current development issue – consult experts if carrying out gas-side fcg assessments - any recent updates?.

Aqueous environments can be very deleterious as regards fcg rates, especially if aggressive radicals are present (chlorides, sulphates, etc.). The result can be stress corrosion cracking (SCC) or corrosion-fatigue. This is a big topic, and of considerable significance for BE – but we will not go into details here (I wouldn't be competent to do so). The shocking truth, though, is that even highly pure, de-oxygenated water can be seriously deleterious to some materials (austenitics) in certain circumstances.

Qu.: What is meant by the Fatigue Crack Growth Threshold?

Fatigue crack growth laws like the Paris Law $\frac{da}{dN} = C\Delta K^m$ do not apply for arbitrarily small stress intensity factor ranges. In fact, below a certain stress intensity factor range there is no fatigue crack growth at all. The ΔK which is required to produce non-zero growth is the fatigue crack growth threshold, ΔK_{th} or ΔK_0.

Qu.: Why is there a fatigue crack growth threshold?

Suppose there were no threshold, i.e., suppose that fatigue crack growth occurred at some level for arbitrarily small ΔK. Furthermore, suppose the Paris law remained applicable.

Consider a 10mm crack undergoing vibration fatigue at a frequency of 100Hz but with a paltry stress range of only 1 MPa. Using the growth law parameters for 316 weld (above) we would conclude on the basis of the Paris law that the growth would be in the order of tens of mm per year, i.e.,

$$100Hz \times 3.1E7 \text{ seconds} \times 9.1E-10 \times \left(\text{compliance} \times 1 \times \sqrt{\pi \cdot 0.01m}\right)^3 \text{ metres} \sim \text{tens of mm}$$

Hence a trivial level of vibration, such as is common in engineering structures, would be enough to fail a cracked structure very quickly.

Alternatively, consider a structure which is nominally defect-free. In truth, real engineering materials always contain flaws at the microscopic, sub-grain-size scale. Suppose our material contains microscopic defects 10 microns in size, and that these behave roughly like cracks. What stress range would be required to cause these flaws to propagate to failure in one year due to vibration at 100Hz, assuming a compliance factor of ~2? The answer is a stress range of order 14 MPa:-

$$100Hz \times 3.1E7 \text{ seconds} \times 9.1E-10 \times \left(2 \times 14 \times \sqrt{\pi \cdot 1.E-5m}\right)^3 \text{ metres} \sim \text{ten mm}$$

A vibration stress range of 14 MPa is rather larger than one would like, but is still not excessive – and yet the above calculation 'predicts' that even a structure which would normally be regarded as defect-free will fail in about a year. In other words this calculation suggests that the fatigue *endurance* of the material is less than 14 MPa.

Clearly there's something wrong here. What's wrong is that the Paris law does not hold for arbitrarily small ΔK. Both the above examples involve SIF ranges of less than 0.2 MPa√m. We may surmise that ΔK_{th} is greater than 0.2 MPa√m, so that the fatigue crack growth in the above examples would really be zero!

Qu.: OK, but ***why*** is there a fatigue crack growth threshold?

The answer is, "because materials are made of atoms/molecules". There is a minimum quantum of growth. A crack cannot advance by less than one atomic spacing (or, at least, some minimum quantum of growth of roughly that size). The fatigue crack growth threshold is a quantum effect.

This provides us with a way of estimating ΔK_{th}. Let's guess that the Paris Law does hold at just slightly above ΔK_{th}, and that at this SIF range the crack growth per cycle equals the minimum possible, i.e., the inter-atomic spacing. The later is of the order of atomic dimensions, i.e., about two Angstroms ($2x10^{-10}$ m). We can thus estimate ΔK_{th} by putting,

$$C\Delta K_{th}^{m} = 2 \times 10^{-10} \text{ meters} \tag{10}$$

Qu.: How does this estimate compare with measured values of ΔK_{th} ?

Using the C and m parameters from R66 (upper bound fcg) gives,

Material	**C**	**m**	**ΔK_{th} (estimated using Equ.9)**
CMn	1.5×10^{-11}	3	2.4
CMn (ST)	4×10^{-12}	4	2.7
CMV	4.1×10^{-11}	3.5	1.6
316 parent	1.4×10^{-10}	3	1.1
316 weld	9.1×10^{-10}	3	0.6

Gratifyingly, these estimates of ΔK_{th} are pretty good, though they should not be claimed in any safety critical arguments! Threshold data from fatigue tests support $\Delta K_{th} \geq 2\text{MPa}\sqrt{\text{m}}$, with thresholds as low as 2 or 3 MPa√m being typical where there is a fairly large mean stress, whereas low mean stress thresholds range between ~2 and ~9 MPa√m (see R66 Section 10, e.g., Figure 10.10). This supports the contention that the fatigue crack growth threshold has a quantum (atomic) origin.

Since all the ΔK_{th} values exceed the SIF ranges of the above examples of 100Hz vibration we see that the fcg would really be zero for these examples. The fatigue threshold is what saves us from trivial levels of vibration destroying everything.

Qu.: How does environment influence ΔK_{th} ?

R66 notes that in corrosive environments the effective ΔK_{th} can be much lower, values as low as 0.1 MPa√m, or even zero, being recommended in some circumstances. R66 is rightly cautious. Corrosion-fatigue can be a virulent mechanism for rapid crack growth. However, how can such low thresholds be possible? The above argument purports to show that such low thresholds are impossible.

I do not know the answer for sure. But the likely explanation is that these very low values of ΔK_{th} are not true fatigue thresholds. Corrosion fatigue tests will involve both load cycling and also dwells in the corrosive environment. The duration of the dwells will affect the amount of crack growth observed. To isolate the purely cyclic contribution to the growth, as opposed from corrosion, the limit of continuous cycling would be required. It is not clear that the tests will have been interpreted in this way. Moreover, corrosion will probably drive crack growth even without any cycling. In other words, there will be growth even for $\Delta K = 0$, so clearly there is no threshold in this sense. But the mechanism is then purely corrosion, not fatigue. The resolution of the paradox therefore lies in the interpretation of the tests rather than anything fundamental (probably).

The preceding remarks regarding a CO_2 environment are again relevant here.

Qu.: What is the effect of mean stress on ΔK_{th} ?

The threshold ΔK_{th} is quite strongly dependent upon mean stress (or mean SIF). This is analogous to the effect of mean stress in reducing fatigue endurance (for example as in the Goodman diagram, where the reduction in fatigue endurance is proportional to the mean stress as a fraction of the UTS). However, this is not in conflict with the above arguments. The minimum ΔK_{th} values derived above are broadly consistent with the minimum values pertaining to high mean stress levels.

Qu.: How is crack growth under vibration fatigue assessed?

Vibration fatigue, or high cycle fatigue, is a big subject in its own right. It has its own British Energy procedure: R2. So we will not attempt to cover it here.

But the reader should be aware that the naïve depiction of vibration loading given above, as a ‘pure tone’ of fixed frequency and constant amplitude, is not at all typical of real structural vibration. Whilst vibration may often be dominated by a particular frequency, or a small number of discrete frequencies, the amplitude will rarely be constant. More typically, vibration consists of “narrow band random” excitation in which the amplitude varies randomly. Typically a Rayleigh distribution of amplitudes may apply, and this is the basis of the R2 procedure.

Unless the structure in question is in desperate trouble, the usual expectation is that the vast majority of the vibration cycles will lie below threshold. Only a small number of occasional cycles will be above threshold and contribute to growth. Hence, the form of the distribution function assumed (e.g., Rayleigh) is crucial since the tail of the distribution determines the implications for the structure. An analogy is with coastal erosion, where normal waves do very little damage. Virtually all the damage is done by storm surges. For more on vibration fatigue consult R2.