Tutorial Session 11: T72S01- Slip Line Field Theory

Last Update: 4/5/2014

Slip line field theory: The hyperbolic nature of the perfect plasticity equations in plane strain: Cauchy surfaces; Example slip line fields for bent bars, sharp cracks and blunt cracks: the relevance to fracture (constraint elevates hydrostatic stress).

This session is beyond SQEP requirements.

Everything in this session relates to plane strain and regions of plastic flow in a perfectly plastic material (no hardening)

Qu.: What is the most general state of stress in plane strain conditions when plastic flow is occurring?

Recall this was covered in http://rickbradford.co.uk/T72S01TutorialNotes8b.pdf. The answer is "an arbitrary hydrostatic stress plus an in-plane shear at the current yield level". Calling the shear yield stress τ_0, and assuming the x,y axes are aligned with the principal axes, the most general plane strain stress state in regions undergoing plastic flow is,

$$\sigma_H \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \tau_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (1)$$

This is an arbitrary hydrostatic stress, σ_H, plus a pure shear at yield magnitude. If the material has been work hardened by the accumulation of plastic strain, then τ_0 is the current shear yield stress after hardening. Rotating to a coordinate system at $\pm 45^o$ to the principal axes in the x-y plane this becomes,

$$\sigma_H \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \pm \tau_0 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (2)$$

Recall that τ_0 in terms of the tensile yield stress, is $\tau_0 = \frac{\sigma_0}{2}$ for a Tresca material, or, $\tau_0 = \frac{\sigma_0}{\sqrt{3}}$ for a Mises material.

Note that this applies only if the region is *currently* undergoing plastic flow. It does not apply if the plastic strains are historical. If stresses have been reversed so that plastic flow is no longer occurring, all bets are off.

Qu.: How is this stress state described in an arbitrary x,y coordinate system?

Alternatively we can define an arbitrary coordinate system by rotating clockwise by an angle ϕ starting from the coordinate system in which the stress looks like a hydrostatic stress plus a pure shear, i.e., starting from the situation described by (2). Using the expressions for the stresses with respect to the rotated coordinate system given in http://rickbradford.co.uk/T72S01TutorialNotes5.pdf we find,

$$\sigma''_x = \sigma_H - \tau_0 \sin 2\phi; \quad \sigma''_y = \sigma_H + \tau_0 \sin 2\phi; \quad \tau'' = \tau_0 \cos 2\phi \qquad (3)$$

Or, in matrix notation referred to this x'', y'' coordinate system,

$$\begin{pmatrix} \sigma_H - \tau_0 \sin 2\phi & \tau_0 \cos 2\phi & 0 \\ \tau_0 \cos 2\phi & \sigma_H + \tau_0 \sin 2\phi & 0 \\ 0 & 0 & \sigma_H \end{pmatrix} \quad (4)$$

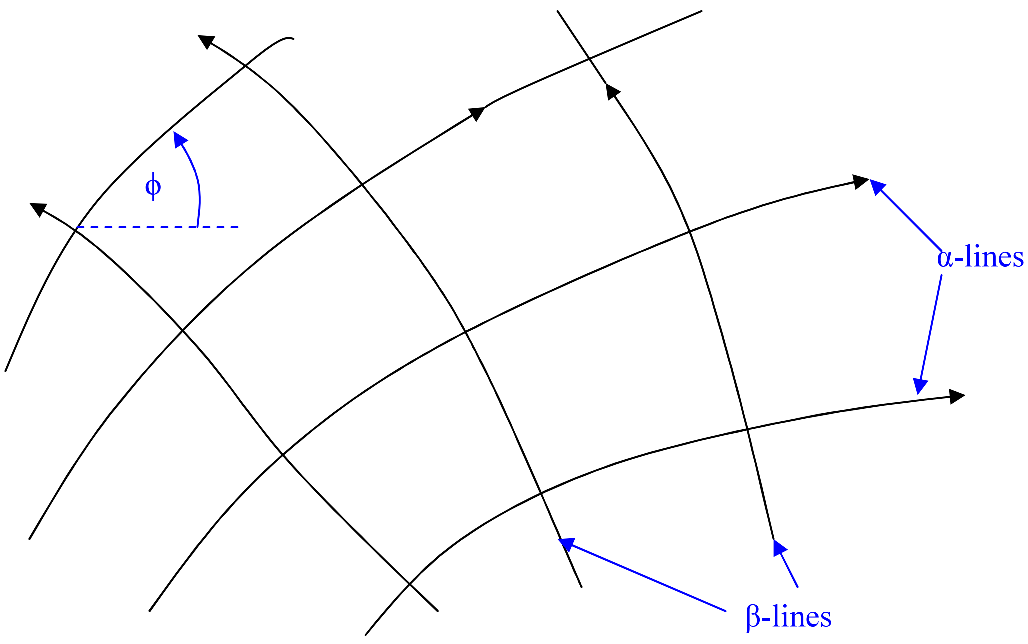
Qu.: How many free parameters are there in Equs.(3,4)?

There are two free parameters in equs.(3) or (4), i.e., the hydrostatic stress, σ_H, and the angle, ϕ. (The yield strength is fixed by the material). If we adopt some arbitrary Cartesian coordinate system, say the above x'', y'' system, then ϕ is the (anti-clockwise) angle which the "axes of shear" make with this system. From here on we will drop the dashes and just refer to this as our x, y system.

This means that, from point to point in the body, the hydrostatic stress, σ_H, and the angle, ϕ, will vary in general. A complete description of the state of stress everywhere within the plastically deforming region is provided by specifying σ_H and ϕ as a function of position.

Qu.: A simpler way of looking at it: The (α, β) "axes of shear"

A simpler way of looking at the general state of stress, (4), is to regard it as pure shear yield plus some hydrostatic stress, as given by (2), but such that the "axes of shear" vary from place to place. The axes of shear, i.e., the orientation of axes such that the stresses look locally like (2), are referred to as the (α, β) system. The orientation of this system changes from place to place, but (α, β) are everywhere perpendicular. So **the complete (α, β) system is an orthogonal curvilinear system**, like this…



Such an (α, β) curvilinear system is called a "**<u>Hencky net</u>**".

Qu.: What constraint does equilibrium impose?

A wonderful simplification occurs when we formulate the equations of equilibrium with respect to this (α, β) "axes of shear" system. Recall that in the Cartesian system the equations of equilibrium in 2D are,

$$\sigma_{x,x} + \tau_{xy,y} = 0 \qquad \text{and} \qquad \sigma_{y,y} + \tau_{xy,x} = 0 \tag{5}$$

Substituting for the stress components from (4) these equations become,

$$\sigma_{H,x} - 2\tau_0 \left[\phi_{,x} \cos 2\phi + \phi_{,y} \sin 2\phi\right] = 0 \quad \text{and} \quad \sigma_{H,y} - 2\tau_0 \left[\phi_{,x} \sin 2\phi - \phi_{,y} \cos 2\phi\right] = 0 \tag{6}$$

The simplification comes about by expressing these equations as changes in the hydrostatic stress along the α or β lines. For example, because an increment along an α line consists of an increment $dx = \cos\phi \cdot d\alpha$ along the x-axis plus an increment $dy = \sin\phi \cdot d\alpha$ along the y-axis, the change in the hydrostatic stress due to an increment along an α line is,

$$d\sigma_H = \sigma_{H,x} \cos\phi \cdot d\alpha + \sigma_{H,y} \sin\phi \cdot d\alpha \tag{7}$$

Substituting from (6) this becomes,

$$\begin{aligned} d\sigma_H &= 2\tau_0 \left[\phi_{,x} \cos 2\phi + \phi_{,y} \sin 2\phi\right] \cos\phi \cdot d\alpha + 2\tau_0 \left[\phi_{,x} \sin 2\phi - \phi_{,y} \cos 2\phi\right] \sin\phi \cdot d\alpha \\ &= 2\tau_0 \left\{\left[\cos\phi \cos 2\phi + \sin\phi \sin 2\phi\right]\phi_{,x} + \left[\cos\phi \sin 2\phi - \sin\phi \cos 2\phi\right]\phi_{,y}\right\} d\alpha \\ &= 2\tau_0 \left\{\cos\phi \cdot \phi_{,x} + \sin\phi \cdot \phi_{,y}\right\} d\alpha = 2\tau_0 d\phi \end{aligned} \tag{8}$$

where the trigonometric identities,

$$\sin(A + B) = \sin A \cos B + \sin B \cos A; \quad \cos(A + B) = \cos A \cos B - \sin A \sin B \tag{9}$$

have been used.

The last step in (8) follows from an equation like (7) which gives the total change in the angle ϕ when we move a distance $d\alpha$ along the α line in terms of the partial derivatives wrt x and y. Writing (8) without the intermediate steps we see that we have simply,

α lines: $$d\sigma_H = 2\tau_0 d\phi \tag{10}$$

Hence, the change in the hydrostatic stress when we move along an α line is simply twice the shear yield stress times the change in the orientation of the α line, $d\phi$.

Following the same derivation through for the β lines gives,

$$d\sigma_H = -\sigma_{H,x} \sin\phi \cdot d\alpha + \sigma_{H,y} \cos\phi \cdot d\alpha \tag{11}$$

$$\begin{aligned} d\sigma_H &= -2\tau_0 \left[\phi_{,x} \cos 2\phi + \phi_{,y} \sin 2\phi\right] \sin\phi \cdot d\alpha + 2\tau_0 \left[\phi_{,x} \sin 2\phi - \phi_{,y} \cos 2\phi\right] \cos\phi \cdot d\alpha \\ &= 2\tau_0 \left\{\left[-\sin\phi \cos 2\phi + \cos\phi \sin 2\phi\right]\phi_{,x} + \left[-\sin\phi \sin 2\phi - \cos\phi \cos 2\phi\right]\phi_{,y}\right\} d\alpha \\ &= 2\tau_0 \left\{\sin\phi \cdot \phi_{,x} - \cos\phi \cdot \phi_{,y}\right\} d\alpha = -2\tau_0 d\phi \end{aligned} \tag{12}$$

Hence, the change in the hydrostatic stress when we move along a β line is simply *minus* twice the shear yield stress times the change in the orientation of the β line, $d\phi$, i.e.,

β lines: $$d\sigma_H = -2\tau_0 d\phi \tag{13}$$

The simple equations (10) and (13) are the complete expression of equilibrium (in plane strain and in regions where plastic flow is occurring).

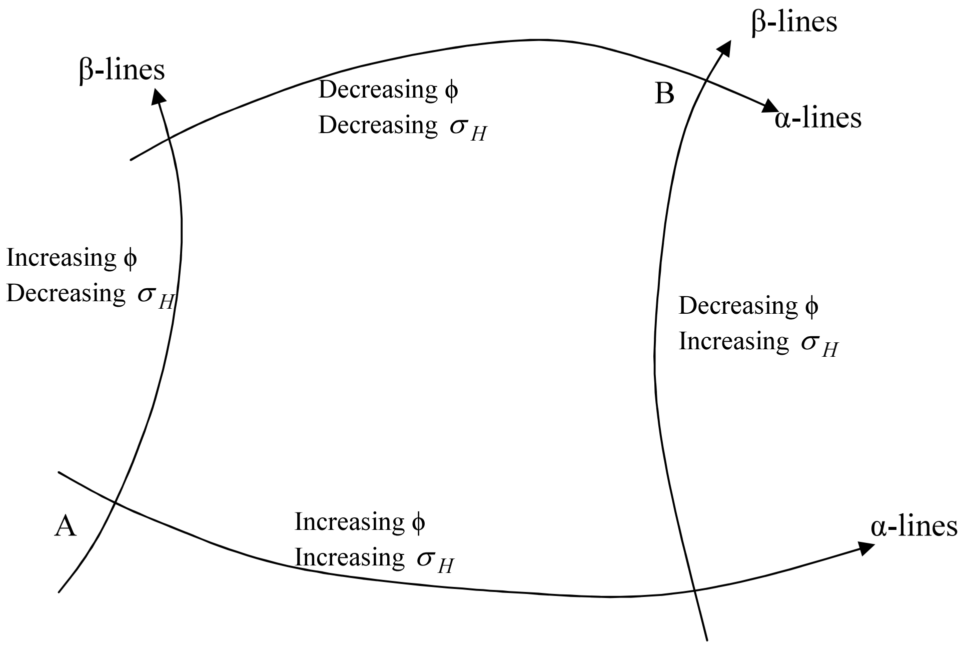
Qu.: What do equations (10) and (13) tell us?

If we are given the geometry of the (α, β) Hencky net, then knowing the hydrostatic stress at any single point means that we can simply read off the hydrostatic stress anywhere else from the changes in orientation of the (α, β) lines which connect the point to the known point. Given the Hencky net, this means we know everything – since the only other degree of freedom is ϕ, which is what the geometry of the net gives us.

Qu.: Can any orthogonal curvilinear system be a Hencky net?

No.

We can travel from one point to another on the net by many different routes, but they must all agree as regards the hydrostatic stress which results. An example of an impossible net is,



Going from A to B in different directions results in an increase in the hydrostatic stress one way, but a decrease the other way – clearly impossible.

Qu.: Are there simple rules to see if a proposed Hencky net is possible?

Yes. They are grandly known as "Hencky's theorems". They all result from the requirement that different routes around the net must give the same hydrostatic stress. Considering the net illustrated overleaf, uniqueness of the hydrostatic stress clearly requires,

$$(\phi_B - \phi_A) - (\phi_C - \phi_B) = -(\phi_D - \phi_A) + (\phi_C - \phi_D) \qquad (14)$$

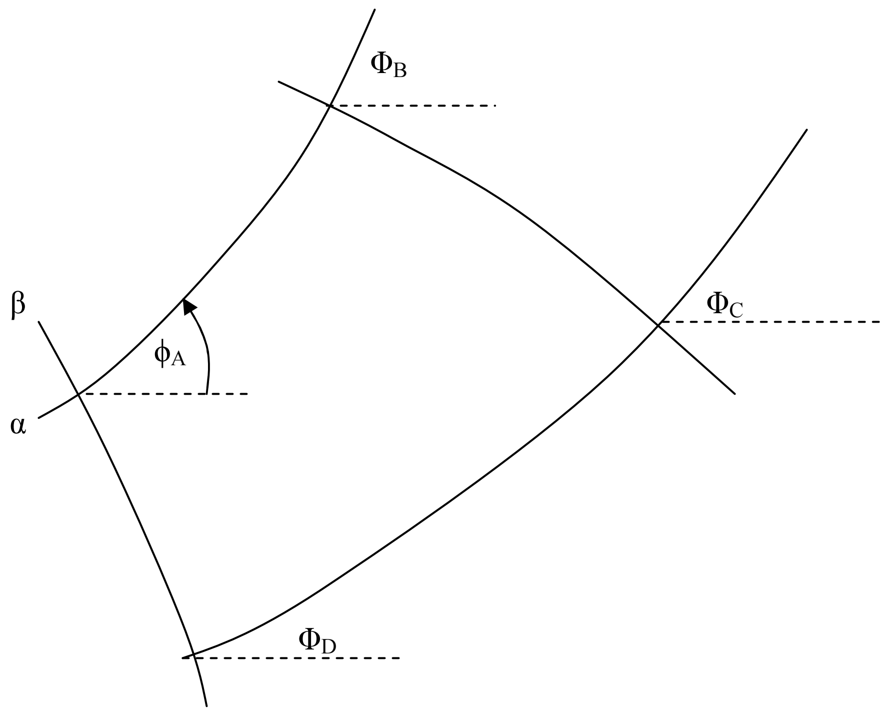
which gives,

$$\phi_A + \phi_C = \phi_B + \phi_D \qquad (15)$$

Alternatively we can write this as,

$$\phi_B - \phi_A = \phi_C - \phi_D \qquad \text{or} \qquad \phi_A - \phi_D = \phi_B - \phi_C \qquad (16)$$

Interpreted geometrically these relations imply…



Hencky's Theorems:-

The angle subtended by a pair of α lines where they cross a β line is the same for all β lines [$\phi_A - \phi_D = \phi_B - \phi_C$].

The change of orientation of an α line between a pair of β lines is the same for all α lines [$\phi_B - \phi_A = \phi_C - \phi_D$].

Corollary: If an α line is straight between a pair of β lines then all α lines are straight between these β lines.

Qu.: What is the boundary condition at a free planar surface?

Say the free surface is a plane with normal in the y direction. Then the y stress and the xy shear stress are zero. From (3) it follows that the (α, β) lines must be at an angle $\phi = \pm 45^o$ to the surface, resulting in,

$$\sigma_x = \sigma_H \mp \tau_0 ; \; \sigma_y = \sigma_H \pm \tau_0 = 0 ; \qquad \tau = 0 \qquad (17)$$

So the hydrostatic stress at the free surface must be $\sigma_H = \mp \tau_0$, and the x-stress is $\sigma_x = 2\sigma_H = \mp 2\tau_0$.

Qu.: What is the boundary condition at a planar surface with an applied y-stress?

The y-direction is the normal to the surface, and the xy shear is again assumed to be zero. Hence, the (α, β) lines must again be at an angle $\phi = \pm 45^o$ to the surface. We now have,

$$\sigma_x = \sigma_H \mp \tau_0 ; \; \sigma_y^{applied} = \sigma_H \pm \tau_0 ; \qquad \tau = 0 \tag{18}$$

So the hydrostatic stress at the loaded surface must be $\sigma_H = \sigma_y^{applied} \mp \tau_0$, and the x-stress is $\sigma_x = \sigma_y^{applied} \mp 2\tau_0$.

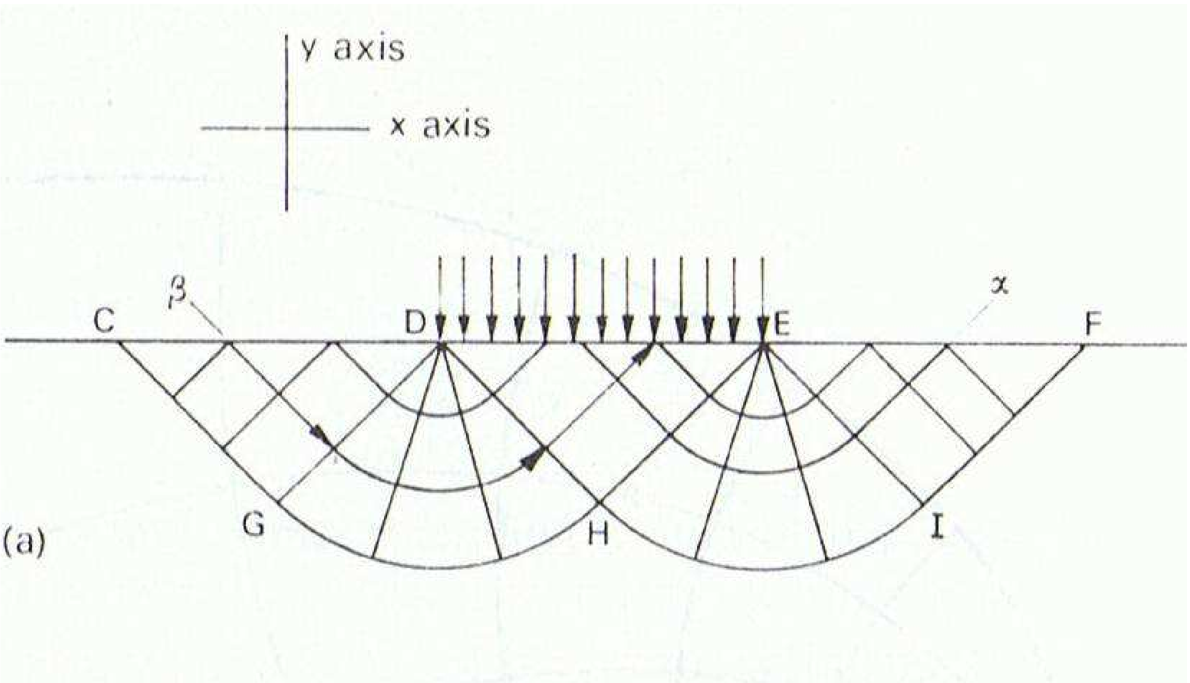
Qu.: What does the Hencky net look like near a shear-free planar surface?

Consider either a free planar surface, or a planar surface which has a load applied in its normal direction but with no applied shear. The (α, β) lines must be at an angle $\phi = \pm 45^o$ to the surface. Consider two points on the surface. How can the α line through one point be extended to cross the β line through the other point? Since they must always cross at right angles, both the α and β lines must rotate by the same angle. But this means that the hydrostatic stress will increase along one and decrease along the other. But both points on the plane boundary start with the same hydrostatic stress (in general, $\sigma_H = \sigma_y^{applied} \mp \tau_0$). So we get a contradiction which is only avoided if, in fact, the (α, β) lines emanating from the plane boundary are straight. Hence,

The Hencky net near a shear-free plane boundary consists of straight (α, β) lines which are at 45-degrees to the boundary

Example Problem (1): Indentation

Consider a semi-infinite slab subject to a flat indenter, thus,



The indenter lies between D and E, applying some compressive y-stress. We are assuming a well-lubricated indenter, so the shear over DE is zero. Hence, over each of the regions CD, DE and EF the net will be composed of straight lines at 45^o to the surface, as shown. The part of the net controlled by the surface condition at DE expires at the point H. Similarly, the region of straight-line net controlled by the free surface regions CD and EF expire at the points G and I.

Within the regions CDG and EFI the hydrostatic stress is constant and equal to $\sigma_H = \mp\tau_0$ [see after equ.(17)]. Since we are clearly dealing with a compressive situation we can assume $\sigma_H = -\tau_0$, $\sigma_x = -2\tau_0$, $\sigma_y = 0$ in this case in these regions.

Within region DEH the hydrostatic stress is constant and equal to $\sigma_H = \sigma_y^{applied} \mp \tau_0$. The + sign must apply in this case, because the magnitude of the hydrostatic stress must be less than the magnitude of the applied y-stress, which is negative. So $\sigma_H = \sigma_y^{applied} + \tau_0 < 0$. Somehow this region must be joined to the outer regions in such a manner that the hydrostatic stress changes. It is clear that the (α, β) lines rotate by $\pi/2$ between the inner and outer constant stress regions, so that the change in hydrostatic stress has *magnitude* $2|\Delta\phi|\tau_0 = \pi\tau_0$. This change is in the sense to make the hydrostatic stress in DEH larger in magnitude (more compressive) than in EFI, so that,

$$\sigma_H(DEH) = \sigma_y^{applied} + \tau_0 = \sigma_H(EFI) - \pi\tau_0 = -\tau_0 - \pi\tau_0 = -(1+\pi)\tau_0 \qquad (19)$$

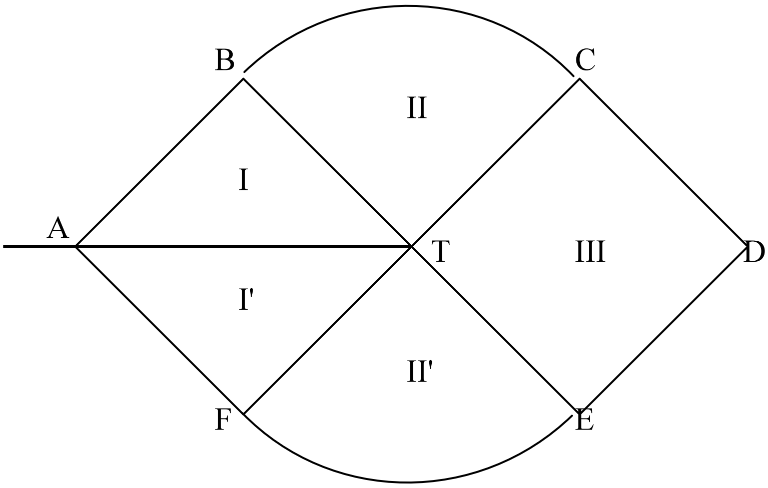
Hennce, $$\sigma_y^{applied} = -(2+\pi)\tau_0 = 2.57\sigma_0(Tresca) = -2.97\sigma_0(Mises) \qquad (20)$$

and, $$\sigma_x = \sigma_y^{applied} + 2\tau_0 = -\pi\tau_0 = 1.57\sigma_0(Tresca) = -1.81\sigma_0(Mises) \qquad (21)$$

(20) shows that the stress required to indent the surface of a semi-infinite slab is ~3 times the uniaxial yield strength (for a Mises material). Recall that our estimate based on the upper bound theorem, http://rickbradford.co.uk/T72S01TutorialNotes10.pdf, was $2.83\sigma_0(Tresca) = -3.27\sigma_0(Mises)$, so reasonably close - and the upper bound is the larger (of course).

The solution also shows that the plastic distortion extends a distance equal to the indenter width on both sides of the indenter. (Strictly we are assuming here that the slip line Hencky net is also indicative of the displacement field. This is shown below).

Example Problem (2): The Mode I Crack Tip Field



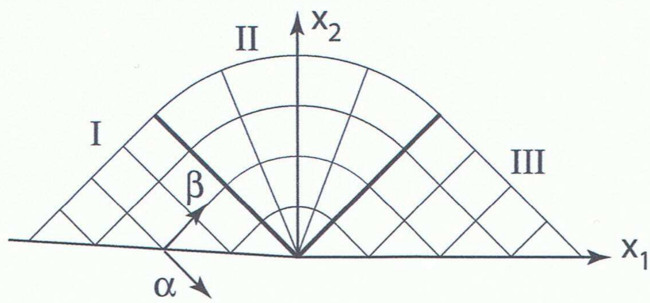


Figure 7.22: Prandtl slip line field for plane strain crack in rigid plastic material. Stress solution from HRR as $n \to 0$ field agrees with slip line field.

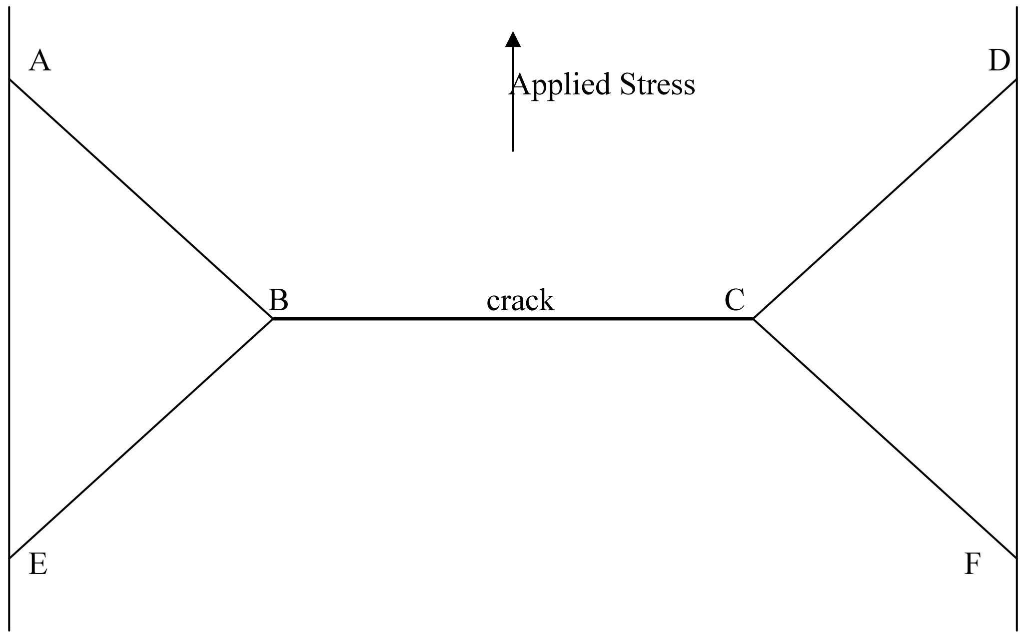
Subject to Mode I tension, the free crack faces, AT, will have hydrostatic stress $\sigma_H = \tau_0$ and x-stress $\sigma_x = 2\tau_0$, but the y-stress and the xy shear will be zero. These stresses are uniform throughout regions ABT and AFT. The hydrostatic stress increases by $2\Delta\phi\tau_0 = \pi\tau_0$ as we rotate from TB to TC into region TCDE, where the hydrostatic stress must therefore be $\sigma_H = (1+\pi)\tau_0$. The xy shear is zero in TCDE, so we get [from $\sigma_H = \sigma_y - \tau_0$ and $\sigma_x = \sigma_y - 2\tau_0$, below (18)] that $\sigma_y = (2+\pi)\tau_0$ and $\sigma_x = \pi\tau_0$. So the Mode I crack problem is very similar to the indenter, but with reversed sign of all stresses.

The important qualitative features of this slip line field solution are,

- Shear straining is confined to the "centred fans" of regions TBC and TEF. This corresponds to the regions of large Mises stress seen in the LEFM (or HRR) crack tip field solutions. This explains why the yield zones around a Mode I crack tip lie at ~90° to the crack line. See answer to homework, http://rickbradford.co.uk/T73S02TutorialHomework13.pdf
- The hydrostatic stress, and the y-stress, ahead of the crack tip, in region TCDE, are very large. The maximum principal stress is $\sigma_y = (2+\pi)\tau_0 \approx 3\sigma_0(Mises)$. This illustrates the high constraint ahead of a Mode I crack in plane strain – again consistent with what we found from the full HRR solution.

In deriving the solution for Example 2 it was assumed that it need only respect the boundary conditions and symmetry conditions applying near the crack tip. In other words, it has been assumed that the boundary conditions applicable to the body as a whole do not interfere with this solution. This is just another way of saying that it has been assumed that yielding is confined to a region around the crack tip and does not extend to the edges of the structure. The next example shows what can happen when the reverse is true – when we get general yielding across the whole section of the structure.

Example Problem (3): The Low Constraint of a Centre-Cracked Plate in General Yield



Regions ABE and CDF have constant stress. The equivalent stress equals the yield stress, of course. The hydrostatic stress is $\sigma_H = \tau_0$, and the y-stress is $\sigma_y = 2\tau_0$ and the x-stress zero. These results follow from the free surface boundaries AE and DF [see after (17)]. Note that this result reproduces the answer to the homework from session 10 using the upper bound theorem: for a Tresca material the y-stress on the ligament at collapse is $\sigma_y = 2\tau_0 = \sigma_0$, whereas for a Mises material it is $\sigma_y = 2\tau_0 = 2\sigma_0 / \sqrt{3}$.

The above centre-cracked plate solution only applies when a load sufficient to cause general yielding has been applied, i.e., a load of $2\tau_0 t(w-a)$, where *t* is the plate thickness, 2*a* the crack length and 2*w* the plate width.

The hydrostatic and maximum principal stresses of $\sigma_H = \tau_0$ and $\sigma_y = 2\tau_0$ respectively compare with the corresponding results ahead of a Mode I crack tip with contained yielding of $\sigma_H = (1+\pi)\tau_0$ and $\sigma_y = (2+\pi)\tau_0$. The hydrostatic stress is more than 4 times larger, and the maximum principal stress more than 2.5 times larger, for the case of contained yielding around a Mode I crack tip.

Hence, for lower loads, when yielding is confined to a region around the crack tip only, we can expect the crack tip fields to be like that of example (2) with high levels of constraint (high hydrostatic and high maximum principal stresses). But when the load is increased to the point of general yield, the constraint decreases markedly (reduced hydrostatic and maximum principal stresses). This is an example of how widespread yielding leads to loss of constraint. If a centre cracked plate (CCP) were used as a fracture toughness specimen, it would cease to be valid under widespread yielding for this reason. A CCP is a low constraint geometry.

Note that this implies that...

The hydrostatic and maximum principal stresses near a crack tip ***decrease*** when the load is ***increased*** as general yield is approached *for perfect plasticity and plane strain*

In particular, proportional loading does not apply.

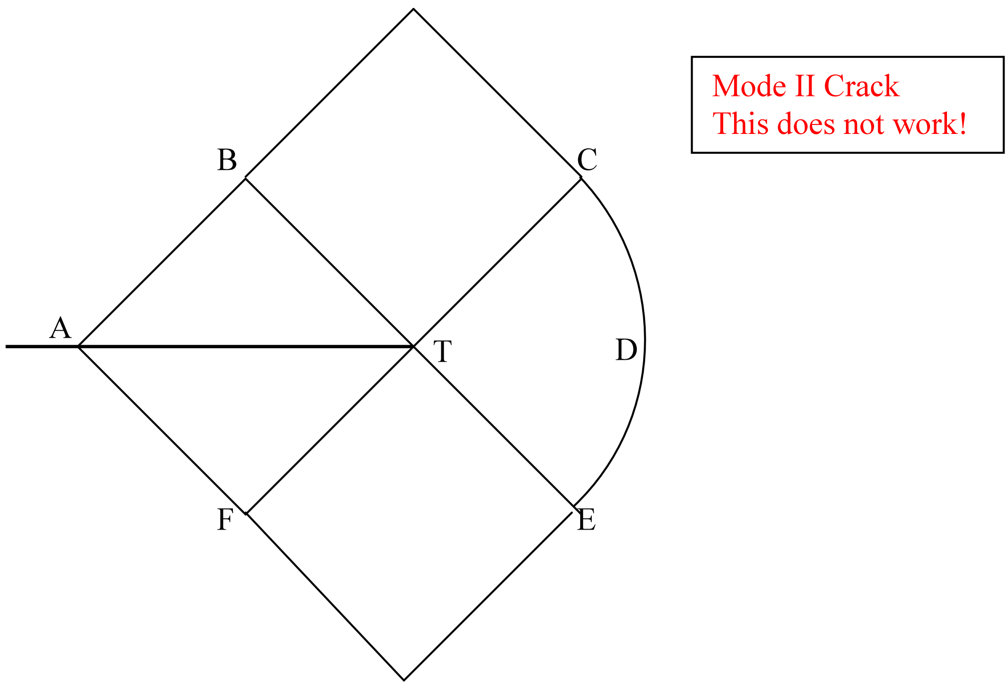
Example Problem (4): The Mode II Crack Tip Field

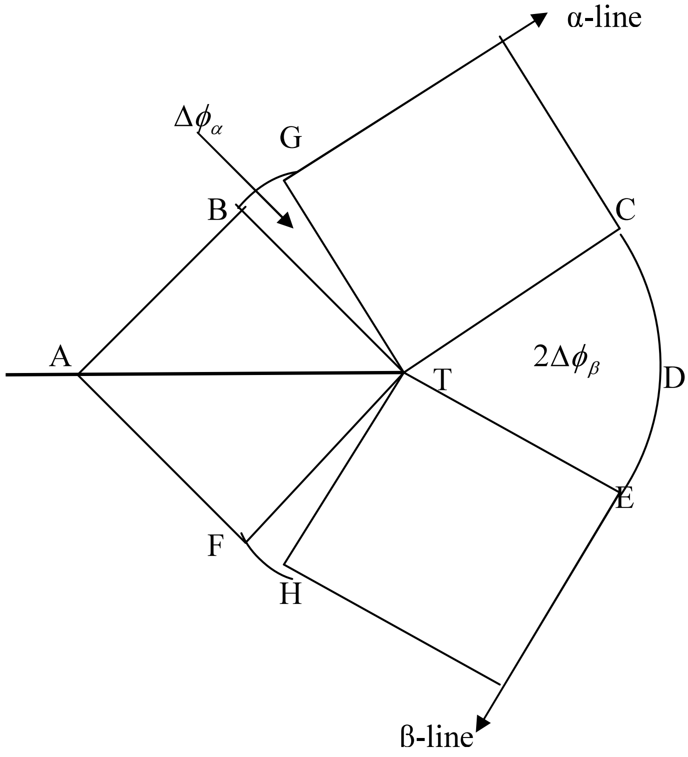
The y-stress and the xy shear must be zero on the crack faces, but now we expect (due to anti-symmetry) that the hydrostatic stress will be $\sigma_H = \tau_0$ on one face and $\sigma_H = -\tau_0$ on the other (the latter is taken to be the top face, consistent with the usual convention). In addition, the anti-symmetry implies that both the x and the y stresses are zero ahead of the crack (on $\theta = 0$), and hence so is the hydrostatic stress, whilst the shear stress is non-zero. Consequently, the x-axis ahead of the crack is an α line. (It doesn't matter if we call it an α or a β line).

How can we achieve a zero hydrostatic stress on $\theta = 0$ given that $\sigma_H = -\tau_0$ on the top crack face? If we try the Hencky net shown below, it does not work because the increase in hydrostatic stress between TC and TD is $2\Delta\phi\tau_0 = \pi\tau_0 / 2$ so that on $\theta = 0$ we get $\sigma_H = \left(\frac{\pi}{2} - 1\right)\tau_0 \neq 0$. This shows that the degree of β line rotation, $\Delta\phi_\beta$, achieved when section TD is reached (starting from AT) must be less than $\pi / 4$. But how can this be done without leaving a gap? The answer is that the rest of the required $\pi / 4$ rotation must be on an α line (see the second trial Henky net, below). Call this α line rotation $\Delta\phi_\alpha$. We thus require,

$$\Delta\phi_\alpha + \Delta\phi_\beta = \frac{\pi}{4} \qquad \text{and} \qquad \sigma_H(\theta = 0) = -\tau_0 + 2(\Delta\phi_\beta - \Delta\phi_\alpha)\tau_0 = 0 \qquad (22)$$

Solving (22) gives $\Delta\phi_\alpha = \frac{\pi}{8} - \frac{1}{4} = 0.1427\,(8.18^o)$, $\Delta\phi_\beta = \frac{\pi}{8} + \frac{1}{4} = 0.6427\,(36.82^o)$. This results in the Hencky net shown overleaf, with a narrow fan 'behind' the crack over which there is a modest decrease in hydrostatic stress, and a larger forward fan which increases the hydrostatic stress, passing through zero on $\theta = 0$.





Mode II Crack
This does work –
but not to scale

A better graphic
is given below

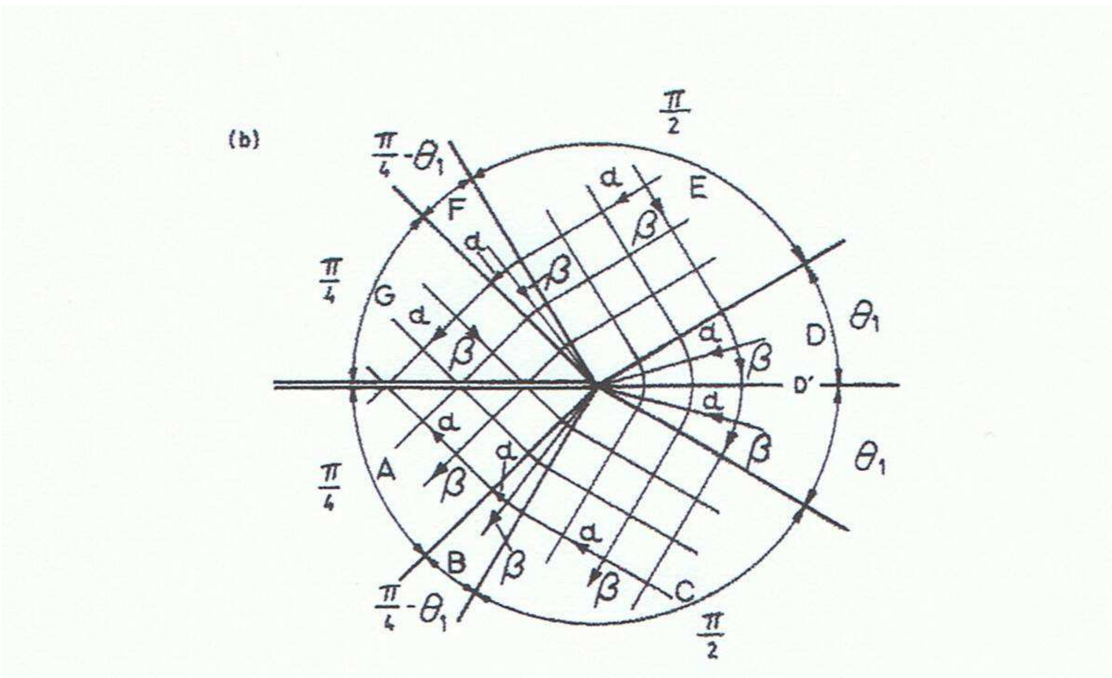


FIG. 5. (a) The elliptical yield contour. (b) The Mode II crack-tip slip-line field.

The resulting stresses are:-

Region ABT: $\sigma_H = -\tau_0$, $\sigma_x = -2\tau_0$, $\sigma_y = 0$, $\tau_{xy} = 0$

Region TCG: $\sigma_H = -1.285\tau_0$, $\sigma_x = -2.245\tau_0$, $\sigma_y = -0.325\tau_0$, $\tau_{xy} = 0.282\tau_0$

Line TD (x axis): $\sigma_H = 0$, $\sigma_x = 0$, $\sigma_y = 0$, $\tau_{xy} = \tau_0$

Region TEH: $\sigma_H = 1.285\tau_0$, $\sigma_x = 2.245\tau_0$, $\sigma_y = 0.325\tau_0$, $\tau_{xy} = -0.282\tau_0$

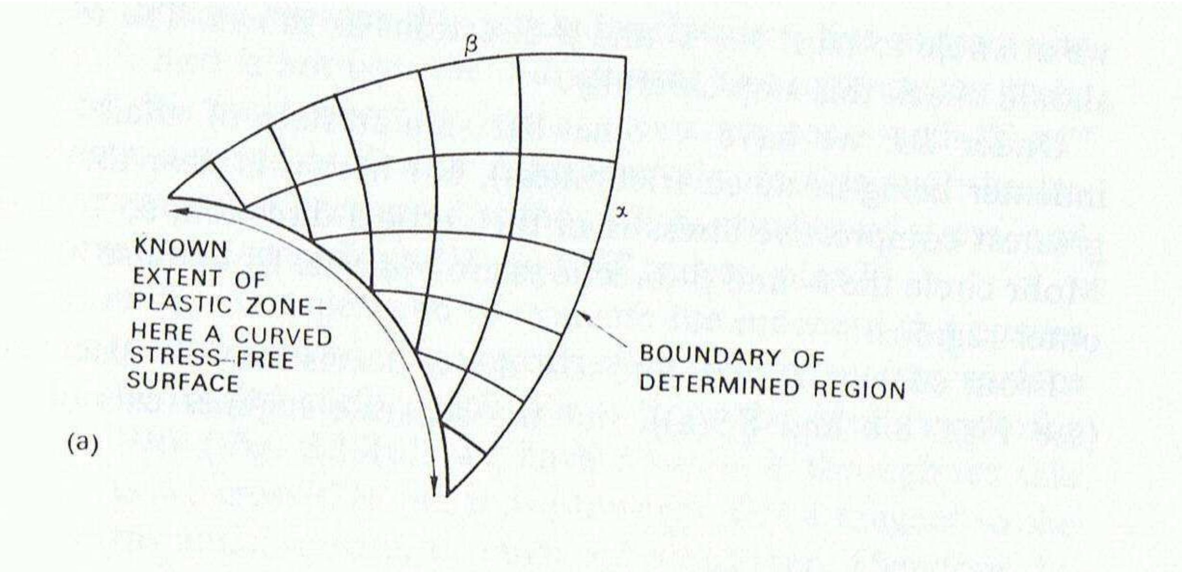
Region AFT: $\sigma_H = +\tau_0$, $\sigma_x = +2\tau_0$, $\sigma_y = 0$, $\tau_{xy} = 0$

So, the regions TCG and TEH, centred a little forward of 90° to the crack plane, have the greatest constraint, but achieve a hydrostatic stress of only $\sigma_H = 1.285\tau_0$ - far smaller than in Mode I – and this is accompanied by only small y and xy shear stresses. The shear stress is only large ahead of the crack, where constraint is vanishingly low. This is consistent with examination of the full crack tip fields (LEFM or HRR) which show that the yield zone extends furthest ahead of the crack, and the hydrostatic stress is greatest either around 90° or behind the crack.

The low constraint of the Mode II plastic field suggests that ductile materials will exhibit higher toughness in Mode II than in Mode I. (In contrast to the conclusion based on the assumption that ductile fracture is energy release rate controlled – which implies equality of Mode I and Mode II toughnesses).

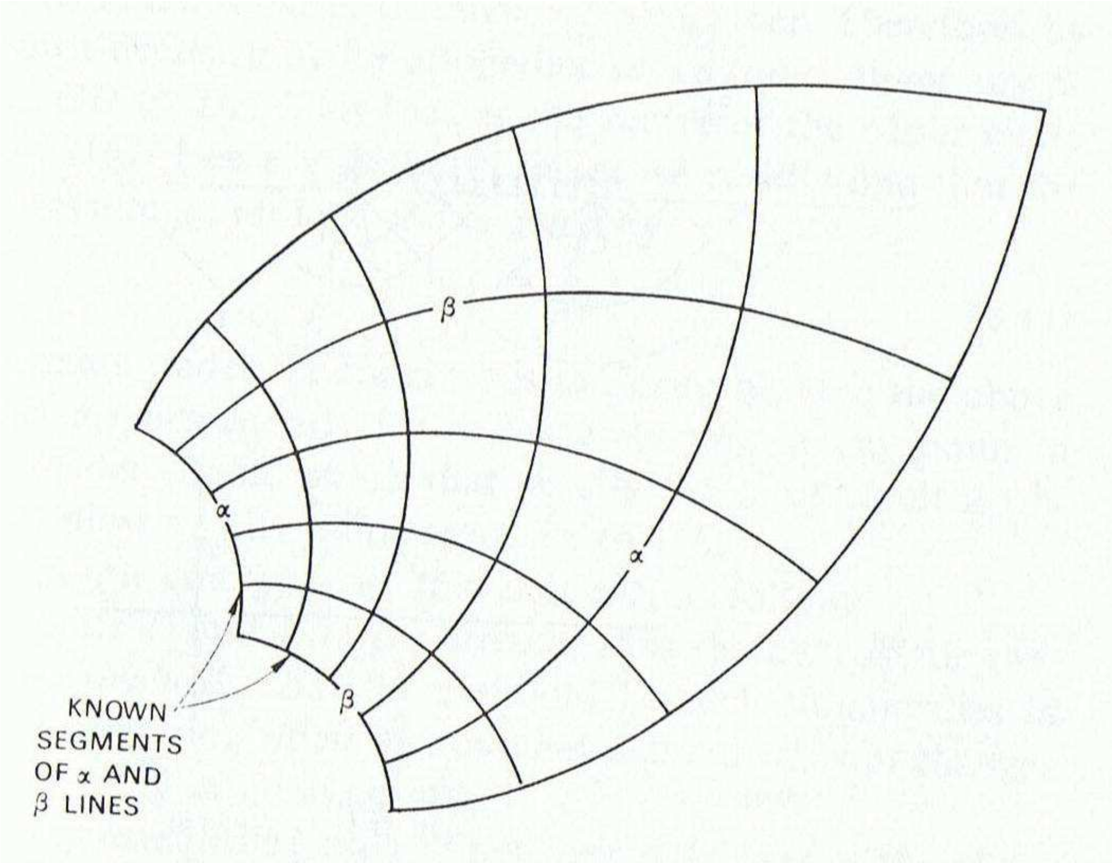
Qu.: All these examples had straight boundaries – what happens when the boundaries are curved?

Even though a free boundary may be curved, the slip lines must still be at 45° to it (so the shear across it is zero). But the slip lines emanating from a curved free surface must also be curved so that they meet at right angles – see illustration below.

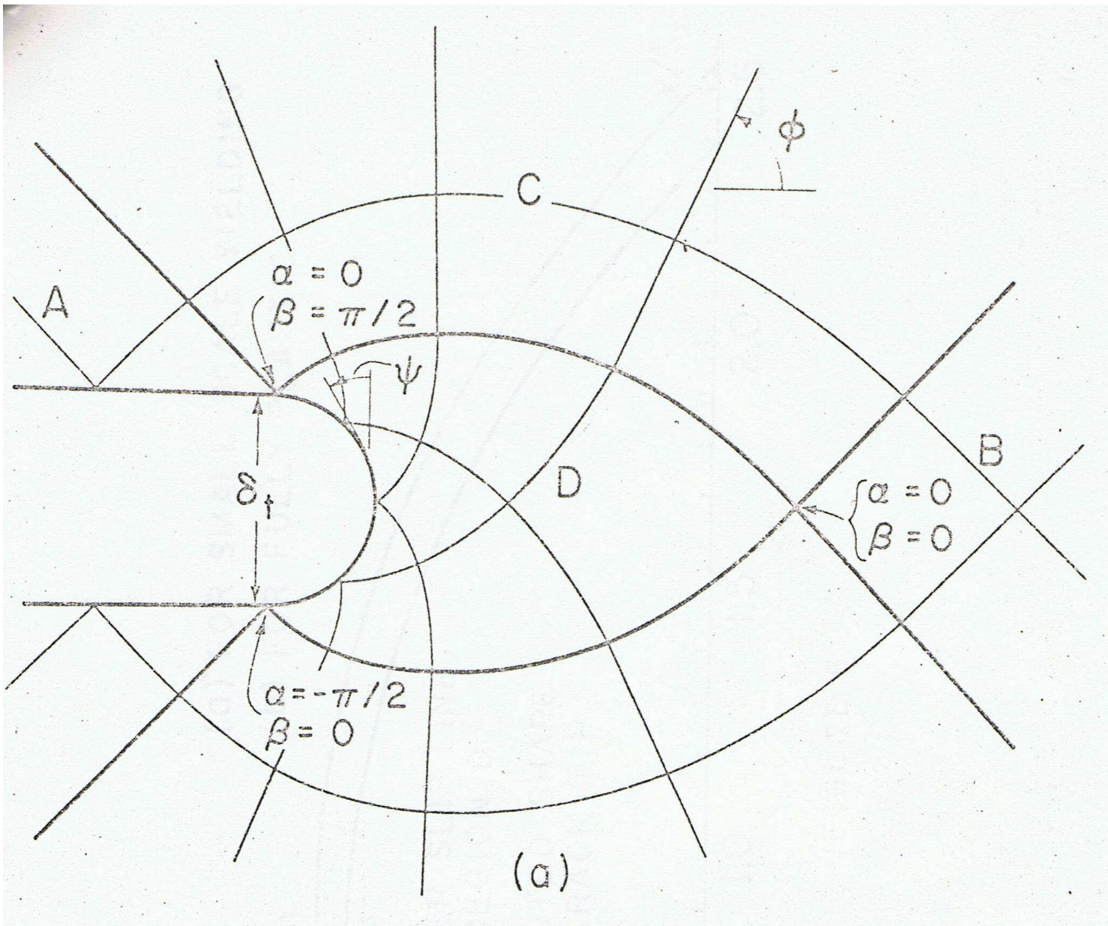


Qu.: If a Hencky net is known up to a certain pair of α and β lines, how can it be continued?

The initial orientation of the lines together with the fact that the extended lines must cross at right angles is enough to define an extension of the net uniquely. But this only applies up to the pair of lines which run from the extremity of the known lines. This is illustrated below.



Example of a Curved Free Boundary Effect: The Blunted Mode I Crack Tip (Rice & Johnson)



The fans which are focussed for a sharp Mode I crack become de-focussed for a blunted crack. The region D, above, is generated which did not exist in the sharp crack case.

A Mixed Mode I and II Crack With Blunted (Flat) Tip (Budden)

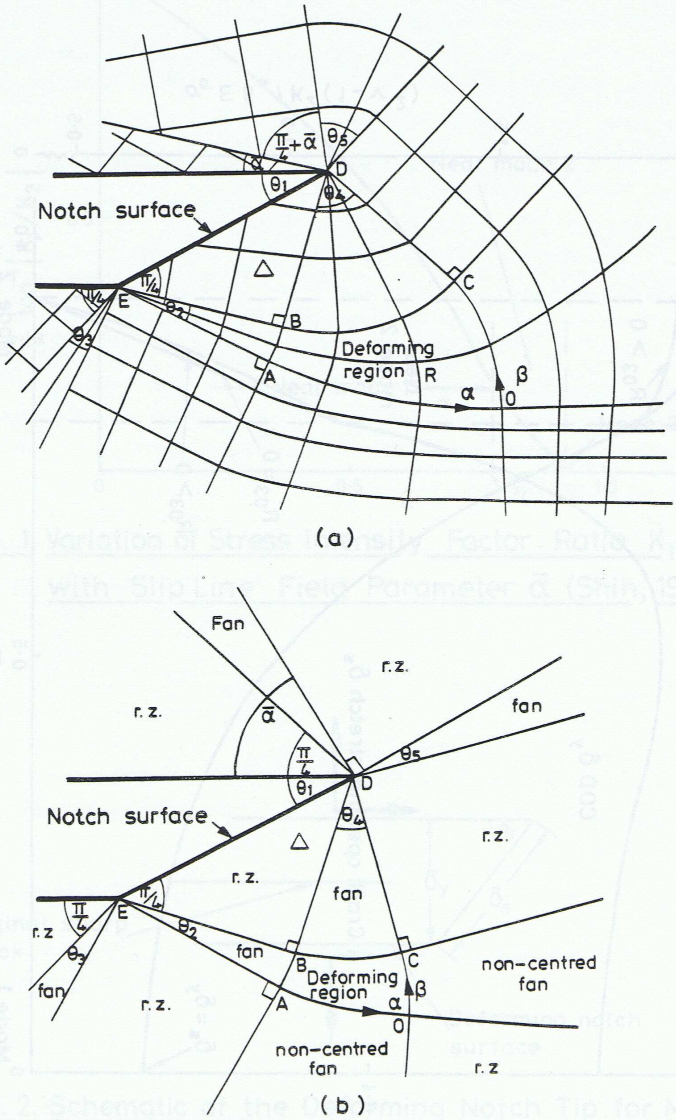


FIG. 4. The Large Deformation Slip Line Field for Mixed Modes 1 and 2 (a) Near Mode 1 ($0 \leq \bar{\alpha} < \pi/4$) (b) Near Mode 2 ($\pi/4 \leq \bar{\alpha} \leq 3\pi/8 - 1/4$) from Budden (1985).

r.z. = Rigid Zone

Qu.: What about displacements / strain rates?

The theory of the slip line field has been developed above entirely in the context of stresses. However, as the terminology "slip lines" implies, these (α, β) nets are also the lines along which slip occurs, and so the slip line solution also tells us about velocities (strain rates). We will not say more about velocities here, although it is proved below that these same (α, β) nets are indeed the lines of slip.

Qu.: Wasn't it rather miraculous that the hydrostatic stress could be expressed as the ordinary differential $d\sigma_H = \pm 2\tau_0 d\phi$ when equilibrium of stresses usually involves partial derivatives?

This brings us to the mathematical origins of the slip lines. The equations of stress analysis for 2D plain strain and perfect plasticity are hyperbolic, and the slip lines are the characteristics of these hyperbolic equations. Rather than develop the general theory of hyperbolic equations and their characteristics, we confine attention to the specific application. Suffice it to say that a characteristic is a curve on which specifying the stresses does not permit the stresses to be found elsewhere.

Qu.: Is this claim consistent with the fact that specifying the stresses on an α-line does not permit the stresses to be deduced off the α-line?

Yes, it is consistent. The examples of Pages 13 and 14 do not contradict this.

The example of Page 13 shows how the stresses within a certain finite region may be found given their value on a given curve. But this illustration only works because the curve chosen is neither an α-line nor a β-line. The size of the region within which the stresses are determined shrinks to zero as the chosen curve is moved closer to either an α-line or a β-line.

The example of Page 14 does show how the stresses within a finite region can be found from boundary conditions specified on an α-line and a β-line. But this only works because *both* an α-line and a β-line are involved. Again, as the length of one or other of this pair of lines is reduced to zero, the size of the region within which the stresses are determined shrinks to zero.

APPENDIX

Stress Characteristics – Proof that they are the (α,β) lines

For both Mises and Tresca materials, the yield condition in plane strain can be written,

$$4\tau^2 + \Delta^2 = 4\tau_0^2, \qquad \text{where, } \Delta = \sigma_y - \sigma_x \tag{23}$$

and τ_0 is the shear yield. Differentiating (23) wrt either *x* or *y* gives,

$$4\tau\frac{\partial \tau}{\partial x} = -\Delta\left(\frac{\partial \sigma_y}{\partial x} - \frac{\partial \sigma_x}{\partial x}\right) \quad \text{and} \quad 4\tau\frac{\partial \tau}{\partial y} = -\Delta\left(\frac{\partial \sigma_y}{\partial y} - \frac{\partial \sigma_x}{\partial y}\right) \tag{24}$$

Substituting (24) into the equilibrium equations, $\sigma_{ij,j} = 0$, gives,

$$4\tau\frac{\partial \sigma_x}{\partial x} - \Delta\left(\frac{\partial \sigma_y}{\partial y} - \frac{\partial \sigma_x}{\partial y}\right) = 0 \quad \text{and} \quad 4\tau\frac{\partial \sigma_y}{\partial y} - \Delta\left(\frac{\partial \sigma_y}{\partial x} - \frac{\partial \sigma_x}{\partial x}\right) = 0 \tag{25}$$

Now consider dx and dy lying along a particular curve. The change in the *x* and *y* stress components along this curve are,

$$d\sigma_x = \frac{\partial \sigma_x}{\partial x}dx + \frac{\partial \sigma_x}{\partial y}dy \quad \text{and} \quad d\sigma_y = \frac{\partial \sigma_y}{\partial x}dx + \frac{\partial \sigma_y}{\partial y}dy \tag{26}$$

The four equations in (25) and (26) can be written in matrix notation as,

$$\begin{pmatrix} dx & dy & 0 & 0 \\ 0 & 0 & dx & dy \\ 4\tau & \Delta & 0 & -\Delta \\ \Delta & 0 & -\Delta & 4\tau \end{pmatrix}\begin{pmatrix} \sigma_{x,x} \\ \sigma_{x,y} \\ \sigma_{y,x} \\ \sigma_{y,y} \end{pmatrix} = \begin{pmatrix} d\sigma_x \\ d\sigma_y \\ 0 \\ 0 \end{pmatrix} \tag{27}$$

The characteristics are those curves such that, given the stresses along the curve (and hence given $d\sigma_x$ and $d\sigma_y$ on the RHS of (27)) we cannot determine the stresses off the curve. But if we knew the four partial derivatives, $\sigma_{i,j}$, then we would know the stresses off the curve by Taylor expansion. Equ.(27) therefore apparently tells us, in effect, how to find the stresses off the curve – by finding the $\sigma_{i,j}$ by inverting the matrix. This is only prevented if the matrix is singular, i.e., its determinant is zero. Hence, the stress characteristics are defined by,

$$\left\|\begin{pmatrix} dx & dy & 0 & 0 \\ 0 & 0 & dx & dy \\ 4\tau & \Delta & 0 & -\Delta \\ \Delta & 0 & -\Delta & 4\tau \end{pmatrix}\right\| = 0 \tag{28}$$

This simplifies to,

$$\Delta\left(\frac{dy}{dx}\right)^2 + 4\tau\frac{dy}{dx} - \Delta = 0 \tag{29}$$

This gives,

$$\frac{dy}{dx} = \frac{2(-\tau \pm \tau_0)}{\Delta} \tag{30}$$

where we have also used (23). We can see that (30) agrees with the previous definition of the (α, β) lines (for example by substituting the expressions from (4) into (30), which shows that either $\frac{dy}{dx} = \tan\phi$ or $\frac{dy}{dx} = -\cot\phi$. More simply, if the shear is zero, then $\Delta = \pm 2\tau_0$ and hence $\frac{dy}{dx} = \pm 1$ and hence the characteristics align with the (α, β) lines, being at $\pm 45^o$. Alternatively, if the stress appears to be pure shear in the *x,y* system, then either the numerator or the denominator of (30) is zero, so that $\frac{dy}{dx} = 0 \text{ or } \infty$, and again the characteristics align with the (α, β) lines, being horizontal & vertical.

We now show that, defined in the same way, the characteristics of the velocity field are identical – thus finally demonstrating that the name "slip lines" is indeed appropriate.

Velocity Characteristics – Proof that they are also the (α,β) lines

The first equation is plastic incompressibility, which in terms of the velocities is,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{31}$$

The principal stress axes are at the following angle wrt the x-axis,

$$\tan 2\theta = -\frac{2\tau}{\Delta} \tag{32}$$

Because an identical tensor diagonalisation is involved in finding the principal axes of the *plastic* strain *increments*, the angle they make with the x-axis is,

$$\tan 2\theta' = \frac{2d\varepsilon_{xy}^p}{d\varepsilon_x^p - d\varepsilon_y^p} = \frac{u_{,y} + v_{,x}}{u_{,x} - v_{,y}} \tag{33}$$

But the plastic flow increments are proportional to the deviatoric stresses, $d\varepsilon_{ij}^p \propto \hat{\sigma}_{ij}$, and if deviatoric stresses are used in (32) it makes no difference to the orientation of the principal stress axes. Consequently, the principal stress and principal plastic strain increment axes are aligned, i.e., $\theta = \theta'$. (Hmmm…this appears to require a Mises flow criterion, but I think the result is more general than that). So equating (32) and (33) gives our second equation,

$$-2\tau\left(u_{,x} - v_{,y}\right) = \Delta\left(u_{,y} + v_{,x}\right) \tag{34}$$

The final two equations are merely the change of *u* and *v* along the curve in question, i.e.,

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \qquad \text{and} \qquad dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy \tag{35}$$

Putting Equs.(31), (34) and (35) together in matrix notation gives,

$$\begin{pmatrix} dx & dy & 0 & 0 \\ 0 & 0 & dx & dy \\ 1 & 0 & 0 & 1 \\ 2\tau & \Delta & \Delta & -2\tau \end{pmatrix} \begin{pmatrix} u_{,x} \\ u_{,y} \\ v_{,x} \\ v_{,y} \end{pmatrix} = \begin{pmatrix} du \\ dv \\ 0 \\ 0 \end{pmatrix} \tag{36}$$

The characteristics are the curves such that, given the velocities along a given curve (hence given the RHS), we cannot deduce the velocities off the curve, i.e., we cannot find the derivatives of the velocities since this would permit a Taylor series to be used to find the velocities off the curve. Hence the characteristics are such that,

$$\left\| \begin{pmatrix} dx & dy & 0 & 0 \\ 0 & 0 & dx & dy \\ 1 & 0 & 0 & 1 \\ 2\tau & \Delta & \Delta & -2\tau \end{pmatrix} \right\| = 0 \tag{37}$$

This reduces to,

$$\Delta\left(\frac{dy}{dx}\right)^2 + 4\tau\frac{dy}{dx} - \Delta = 0 \tag{38}$$

This is identical to (29), and hence has the same solutions, (30). This demonstrates that the velocity characteristics are the same as the stress characteristics. The Hencky net is therefore a field of slip lines.