Schrodinger's Cat Released At Last

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Hawking famously said, "when I hear of Schrodinger's cat, I reach for my gun". I'm with him all the way on that. Whole books have been written on Schrodinger's cat, and innumerable articles. For heavens sake, why? The apparent paradox arises merely from a misunderstanding of quantum mechanics. I can forgive Schrodinger and his contemporaries. They were groping towards an understanding of quantum mechanics with little in the way of helpful signposts. But the resolution of the paradox has been implicit within standard quantum mechanics almost since the problem was first posed. Admittedly, I did not realise this myself until recently, so I should not be too smug. I had always regarded the density matrix as something I could very well do without. Unfortunately, you really cannot. But I run ahead of myself.

Here is Schrodinger's difficulty. Quantum mechanics tells us that the state of a pure quantum system is described by a vector in Hilbert space. The Hilbert space may be spanned by some orthonormal basis vectors, such as those defining states of definite energy. The most general state will be a superposition of such energy states. Consider an unstable (radioactive) nucleus. Simplifying somewhat, we can regard this nucleus as having two energy states available to it, undecayed and decayed, which we write as $|u\rangle$ and $|d\rangle$. If we have not observed the nucleus, it will be in some superposition of these states, $\alpha|u\rangle + \beta|d\rangle$.

Now here is the essence of the problem. Quantum mechanics holds that the nucleus is neither decayed nor undecayed until its state has been observed (or measured, if you prefer). Until then, its state is indeterminate. We now place the poor cat in a box together with a contrivance which activates a poison gas canister if the nucleus decays. This is a perfectly feasible arrangement. The puzzle presented by this setup is that the cat apparently shares the indeterminacy of the nucleus. The cat will be dead if and only if the nucleus has decayed. Since the state of the nucleus is indeterminate, it is also indeterminate (we are told) whether the cat is alive or dead. The full force of this conundrum is felt only if one realises that quantum indeterminacy is absolute.

As regards a genuine quantum system, such as the nucleus, it truly ***is*** in a superposition of the two states. It is not simply that we are ignorant of the state of the nucleus. The nucleus itself does not know what state it is in! Superpositions are essential in quantum mechanics, for example in giving rise to interference effects. Superpositions of states are what give particles their 'wave-like' properties. Superposition is not optional but crucial to the nature of quantum mechanics.

Now suppose we represent the state of the cat as either $|alive\rangle$ or $|dead\rangle$. If we peeked inside the box and discovered the nucleus to still be intact and the cat alive and well, then the state of the nucleus-plus-cat system would appear to be $|u\rangle|alive\rangle$. Conversely, if we peeked inside the box and discovered the nucleus to have decayed and the cat to be dead, then the state of the nucleus-plus-cat system would appear to be $|d\rangle|dead\rangle$. It would appear, therefore, that prior to our observation the nucleus-plus-cat system must have been in a superposed quantum state of the form $\alpha|u\rangle|alive\rangle + \beta|d\rangle|dead\rangle$. Hence, the cat was neither alive nor dead, but in a quantum superposition of aliveness and deadness, a situation which runs counter to our intuition.

But this is just silly. The source of the misunderstanding lies in the absolute smallness of things like atoms and nuclei, and the absolute largeness of cats. The lesson of quantum mechanics is that the states of small quantum systems like atoms can be exactly defined by a small number of integers. Ignoring excited states of the nucleus, the quantum states of an atom can be defined by the quantum numbers n, j, m and s_z for each of its electrons. In principle, though it is less well understood, the state of our unstable nucleus can also be expressed in terms of quantum numbers like n, j, m and s_z for each of its component protons and neutrons. Thus, improving somewhat on our previous description, the state of our radioactive nucleus can be written $\left|\{n_i, j_i, m_i, s_{zi}\}\right\rangle$. In practice our previous description may be quite adequate since there will be some particular state, $\left|\{n_i, j_i, m_i, s_{zi}\}_u\right\rangle$, which defines the particular radionuclide in question, and perhaps just one final state (assuming there is only a single decay mode), $\left|\{n_i, j_i, m_i, s_{zi}\}_d\right\rangle$. So we might as well write these using the shorthand $|u\rangle$ and $|d\rangle$. But it is important that the correct, completely specified, quantum states $\left|\{n_i, j_i, m_i, s_{zi}\}_u\right\rangle$ and $\left|\{n_i, j_i, m_i, s_{zi}\}_d\right\rangle$ lie behind this shorthand. This is because these really are genuine, pure quantum states.

Now tell me, what is the quantum state of the cat? It is entirely fallacious to suppose that a pure quantum state has been defined simply by enclosing the word “alive” inside a ket, thus: “$|alive\rangle$”. This is ***not*** a pure quantum state. It has just been written to look like one. But actually a living cat cannot be in a pure quantum state.

It is worth pausing awhile to consider how many integers would be required to specify a pure quantum state of a cat. A lower bound will suffice. A cat contains of the order of 10^{26} atoms. Let us assume that all these atoms’ nuclei are in their ground state, and not liable to be excited. Let us further assume that all electrons in the atoms will also remain unexcited, except for a small number of the outer valence electrons. These will be involved (or not) in forming chemical bonds with other atoms. How many different states might these valence electrons be in? I do not know – but it suffices to assume just two states are available – bonded or not. This is a huge simplification and will cause our estimate of the number of possible quantum states to be a gross underestimate. However, this convincingly demonstrates that a lower bound to the number of electron states in the cat must be at least 2 raised to the power 10^{26}. In other words, at least 10^{26} binary digits are required to specify the electrons’ state – and really far more than this.

But then there are the possible states of each atom considered as a whole. The atoms will be jiggling about under the influence of the thermal energy which pervades the whole cat. At room temperature the typical thermal energy (kT) is 4 x 10^{-21} J. How many vibration quantum modes will this excite? As a crude estimate treat the atom as a particle in a cubic box of side equal to the diameter of an atom, around 2 Angstrom. The quantum of energy is $\pi^2\hbar^2/2mL^2$. For a carbon atom of mass 2 x 10^{-26} kg this evaluates to E_0 = 7 x 10^{-23} J. The vibrational modes can be described by three positive integers k, p, q so that the energy of this state is $E(k,p,q)) = E_0(k^2 + p^2 + q^2)$. It follows that, for E ~ kT, the typical value of $(k^2 + p^2 + q^2)$ is ~60. Hence, the ‘radius’ in this (k,p,q) parameter space is typically $r \sim \sqrt{k^2 + p^2 + q^2} \sim 8$. The number of states is the volume within, or close to, such a radius, i.e., of order $r^3 \sim 500$. Thus every atom has of order 500 thermal vibration mode quantum states easily accessible to it. All of these 500 states are guaranteed to be excited in some atom given such a large population of atoms as a cat. The number of binary digits required to represent ~500 states is ~9. To specify the vibration state of all the atoms, and the state of the electrons, therefore requires $(9 + 1) \times 10^{26} = 10^{27}$ integers. So, a lower bound for the number of quantum states which the cat could be in is 2 raised to the power 10^{27}. This is a colossal number. It makes the number of protons in the observable universe ($\sim 10^{80}$) look trivially small, as it does the number of photons ($\sim 10^{91}$). The maximum number of elementary computations that could have been carried out anywhere in the universe throughout its entire history ($\sim 10^{123}$) is also trivial in comparison.

So we are agreed that specifying a pure quantum state of a cat is not a practical possibility. But let us suppose that by some stupendous miracle of experimental technique we were able to place a cat into a pure quantum state, whether we knew what that state was or not. I have bad news for you. Your cat would already be dead. Why? Well, what constitutes being alive? Amongst other things a living organism is a seething mass of complex, ceaseless, chemical reactions. These continual chemical reactions are essential to life. They *are* life. This means that the electron states of your cat must be in continual flux if it is to have any chance of being alive. Its state must be constantly changing via a mess of incoherent (thermally driven) reactions. So I'm afraid your miraculous experimental technique was merely a sophisticated way of killing your cat.

Actually there is an even simpler way of seeing that your pure-quantum-state-cat is a dead cat. If the cat is in a pure quantum state then it is not in a thermal state – because a thermal state is a mixture of energy states. But living organisms survive only within a quite restricted range of temperatures. They rather depend upon being in a thermal state. By 'freezing' your cat into a pure quantum state you bring about its death.

Finally, here is yet another way of understanding the pure-quantum-state-cat's demise. Suppose this pure quantum state is an energy eigenstate. These are stationary states. Their only time dependence is through a phase factor, which makes no physical change to the cat. Such a cat is frozen in time. Nothing about it changes. Since life is a dynamic condition, this is synonymous with death.

So we now understand that writing the state of the cat as $|alive\rangle$ is a contradiction, a subterfuge. On the one hand, this notation makes it seem as if we are dealing with a pure quantum state, whilst on the other hand being alive denies this possibility. And since the cat cannot be described by a pure quantum state, an entangled state like $\alpha|u\rangle|alive\rangle + \beta|d\rangle|dead\rangle$ does not arise. Hence there is no reason to attribute weird quantum indeterminacy to the cat. The cat is not simultaneously both alive and dead. It is not in a quantum superposition of states of aliveness and deadness.

So, is the cat alive or dead? Of course we do not know unless we look inside the box. But our lack of knowledge on this subject is just the usual, classical, deterministic lack of knowledge that arises when anything happens beyond the reach of our senses or instruments. At any instant the cat is in a definite state, either alive or dead. We just do not know which. The radioactive nucleus, on the other hand, does indeed start off in a weird quantum superposition. However, whilst the nucleus may have the power to deal death on the cat, but it does not have the power to contaminate the cat with its quantum indeterminacy.

You may, however, still be feeling dissatisfied. There is something too glib about this explanation, perhaps? This feeling of unease probably springs from the fact that we seem to have denied the cat a description in quantum mechanical terms. The essence of our explanation is to veto the possibility of a pure-quantum-state-cat. But if quantum mechanics is the correct description of the world, everything – cats included – must be expressible within its lexicon. Yes indeed. And it is. This brings us to the most important thing – the density matrix.

We have argued that only ‘small’ things can generally be described successfully by pure quantum states, that is by vectors in a Hilbert space. By ‘small’ we mean that their quantum state is defined uniquely by a relatively small number of integers (and generally this *does* mean they are small). So, how are ‘big’ things described in quantum mechanics?

Moreover, one of the objections to a pure-quantum-state-cat was that it could not be in a thermal state. So how are thermal states to be described quantum mechanically?

Finally, our cat turns out to be in an unknown but definite (deterministic) state. How can the combined cat-plus-nucleus system be described in quantum mechanics if one part is in a genuine superposition quantum state and the other is in an ordinary, deterministic-but-unknown state?

The answer to all these questions is the same: the density matrix. The density matrix formalism permits the quantum mechanical (Hilbert space) features to coexist with a deterministic probability distribution of states. In this description there is both ordinary lack of knowledge, underlying which is a definite state, and also true quantum indeterminacy, entanglement and the potential for interference effects, etc.

Consider our cat. Suppose its energy eigenstates are written $\left|E_i\right\rangle$, where we have seen that the index i must run from 1 to $\sim 2^{10^{27}}$. A pure quantum state can be written,

$$\left|\psi_j\right\rangle = \sum_i a_{ji}\left|E_i\right\rangle \tag{1}$$

(There may be any number of different quantum states like this, labelled by $_j$). We have argued that such a state cannot represent a living cat (though it could represent the corpse of a cat at absolute zero). Now consider instead the object,

$$\hat{\rho} = \sum_j p_j\left|\psi_j\right\rangle\left\langle\psi_j\right| \tag{2}$$

This is the density matrix. It is an operator on Hilbert space rather than a vector in Hilbert space. Physically this density matrix represents an ordinary (deterministic) mixture of quantum states, $\left|\psi_j\right\rangle$, such that each state has the probability p_j of being realised. Note that these coefficients are probabilities, as opposed from the probability *amplitudes*, a_i, which appear in (1). The latter are complex numbers and are related to probabilities by “*probability* = $\left|a_i\right|^2$”. In contrast the p_j in (2) are ordinary probabilities, and hence are real numbers in the range [0,1].

The density matrix, (2), is the valid way to express the state of a cat in quantum mechanical terms. But note that it involves an ordinary, deterministic, probability distribution as well as quantum states.

Unfortunately students are still taught that expressions like (1) represent the most general quantum state of a system. This is just wrong. They are only the most general pure quantum state. But the most general quantum description is (2). A description based on (2) is essential for a living cat. It is also essential for any system which is tending towards classical behaviour – or any system, even a single atom, which is in a thermal state. The classical limit of quantum systems can only be understood through (2), the density matrix. If you stick to the description provided by (1) then you are doomed to be forever perplexed that classical systems appear to have retained their quantum weirdness. Of course – by imposing (1) upon your classical system, you have condemned it to remain a superposition, and hence to retain its quantum weirdness. But this is not reality. Reality is (2).

If (2) is the most general description of any physical system, how is a pure quantum state described in these terms? This is simplicity itself. If you wish to describe the pure state $|\psi\rangle$ then the density matrix is just $\hat{\rho} = |\psi\rangle\langle\psi|$. This contains all the information available in $|\psi\rangle$. For example, the expectation value of any observable $\hat{Q}$ in the state $|\psi\rangle$ would normally be written $\langle\psi|\hat{Q}|\psi\rangle$. In terms of the density matrix it is $Tr(\hat{\rho}\hat{Q})$. This is seen as follows, using $|i\rangle$ to represent any orthonormal basis:-

$$Tr(\hat{\rho}\hat{Q}) = \langle i\|\psi\rangle\langle\psi|\hat{Q}|i\rangle = \langle\psi|\hat{Q}|i\rangle\langle i\|\psi\rangle = \langle\psi|\hat{Q}|\psi\rangle \qquad \text{QED.}$$

But in the general case of a mixed state, when the density matrix is given by (2), the expectation value is,

$$Tr(\hat{\rho}\hat{Q}) = \sum_i p_i \langle\psi_i|\hat{Q}|\psi_i\rangle \tag{3}$$

This is a combination of the quantum-mechanical expectation values of the individual pure quantum states, $\langle\psi_i|\hat{Q}|\psi_i\rangle$, and the deterministic expectation values of these over the probability distribution, p_i. The density matrix thus combines the effects of both types of uncertainty.

A thermal state of a system is described by (2) when the probabilities take on the values required for a system in thermal equilibrium, i.e., $p_i \propto \exp\{-E_i / kT\}$. A thermal state is an ordinary (deterministic) mixture of pure quantum states. This is another way of saying that the various pure quantum states, of various energies, combine randomly, i.e., incoherently. This randomness (entropy) is the essence of the thermal state. It is the physical reason why thermal states must be mixed, and not pure states. The entropy is $-\sum_i p_i \log_2 p_i$, where p_i are the (deterministic) probabilities in (2). Thus, a highly mixed state, with many contributing states, has large entropy. On the other hand, the entropy of a pure state is zero (because it has just one state in (2), with p=1).

Note that although pure states like (1) can also be equivalently expressed in terms of a density matrix, (2), the reverse is not true. The general state of a system defined by (2) cannot in general be expressed as a pure quantum state like (1). The density matrix, (2), is a more general description than (1).

The defining feature of a classical system, like a cat, is that it is highly mixed (and hence has large entropy). That is, its density matrix, (2), includes a very large number of quantum states which contribute to a comparable degree.

We have seen that an entangled state like $\alpha|u\rangle|alive\rangle + \beta|d\rangle|dead\rangle$, which was supposed to imply the quantum indeterminacy of the cat, is a fiction because the cat cannot be represented by a Hilbert space vector, i.e. a pure quantum state. Also, we have seen that the density matrix, (2), is the correct quantum description of a cat. The final question is, "how do we represent the combined state of the cat-in-the-box, including the nucleus and the deadly contrivance with the vial of poison?" The complete answer would entail a complete answer to the measurement problem. For that is what Schrodinger's Cat actually is. It is a measuring device, designed to measure the state of the nucleus. A Geiger counter would register the decay of the nucleus by the flickering of a needle, or an audible bleep. The cat does so by dropping dead. There is no difference of physical principle.

A complete discussion of quantum measurement, and its attendant problem of the collapse of the wavepacket, would take us too far astray (see other Notes on this site). But the final outcome is easy to see. Suppose we partition the density matrix for a cat into parts which represent a living cat and the remainder which represent a dead cat. The density matrix for the cat can thus be written,

$$\hat{\rho} = \sum_i p_i^{alive}|\psi_i\rangle\langle\psi_i| + \sum_i p_i^{dead}|\psi_i\rangle\langle\psi_i| \qquad (4)$$

The undecayed state of the nucleus can be represented just as well by the density matrix $|u\rangle\langle u|$ as by the Hilbert space vector $|u\rangle$ itself, and similarly $|d\rangle\langle d|$ is a complete representation of the decayed nuclear state. The first of these is associated with the first term on the RHS of (4), i.e. with a living cat, whereas $|d\rangle\langle d|$ is associated with the second term in (4), i.e. the dead cat. If the nucleus had been in a state $\alpha|u\rangle + \beta|d\rangle$ prior to the introduction of the cat into the box, then the state $|u\rangle\langle u|$ is associated with a probability of $|\alpha|^2$, and $|d\rangle\langle d|$ is associated with a probability of $|\beta|^2$. Thus, the combined cat-plus-nucleus system is represented by the sum of the direct product states,

$$\hat{\rho} = |\alpha|^2 \sum_i p_i^{alive}\left(|\psi_i\rangle\langle\psi_i|\right)\left(|u\rangle\langle u|\right) + |\beta|^2 \sum_i p_i^{dead}\left(|\psi_i\rangle\langle\psi_i|\right)\left(|d\rangle\langle d|\right) \qquad (5)$$

What this reveals is that, not only has the cat no weird quantum superposition properties, but the nucleus is no longer in a quantum superposition either. The wavepacket of the nucleus has collapsed, as it does in any measurement, leaving it in a definite state. We do not know which state, of course – because we have not looked inside the box yet. This latter, purely deterministic, uncertainty is represented in (5) by the sum of the two terms. The probability of the cat being alive is $|\alpha|^2$ and the probability of it being dead is $|\beta|^2$. These are, of course, exactly the probabilities that would be expected from the original quantum state of the nucleus, $\alpha|u\rangle + \beta|d\rangle$. The crucial difference now is that, having interacted with the cat, the measurement of the nucleus's state *has already been made*. The wavepacket has already collapsed *before you look in the box*. The measurement is not made when you happen to look in the box, which would be horribly subjective. A measurement is a physical interaction between the system being measured and the measuring apparatus – in this case a cat. The measurement is made when the interaction takes place. It is a physical event, not some mystical process. Let us hear no talk of your consciousness being responsible for the collapse of the wavepacket, or other similar nonsense.

In short, rather than the nucleus contaminating the cat with its quantum indeterminacy, the opposite happens. The cat contaminates the nucleus with its classical determinacy, thus collapsing its wavepacket (performing a measurement).