Quantum Teleportation

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1. What Is Quantum Teleportation?

Of course, it's a cheat, really. The classical equivalent of what passes for teleportation in the quantum communication fraternity would be "here's an instruction book for how to construct Captain Kirk atom by atom – the rest is up to you". No matter or energy, let alone the whole of Captain Kirk, is actually transported anywhere. All you get is….well, I'm tempted to say "information", but even that is not quite right. What you get is the exact quantum state. To be impressed by this, you first need to appreciate the no-cloning theorem (see Part 9 of these QM Notes, QM9). This says that there is no process that will produce an exact copy of an arbitrary quantum state. But quantum teleportation will provide Bob with an exact replica of Alice's quantum state. How does this get around the no-cloning theorem? Simple – Alice's original of the quantum state is destroyed in the process. So, this is why it is called "teleportation". Bob does not receive a *copy* of the quantum state – because that's impossible. He gets the *original* – and it's been teleported to him. Cool, or what?

The important thing is that Alice does not know herself the quantum state which is to be teleported. Say it is a qubit state, $\alpha|0\rangle + \beta|1\rangle$. Alice does not know α and β. Otherwise, of course, it would be trivial.

Does the quantum state get transmitted to Bob transluminally? No. It's true that entangled quantum states do genuinely exhibit non-local characteristics, the spooky action-at-a-distance which perturbed Einstein so. And there is a key moment when Alice performs a measurement and the $\Re$-process provides Bob instantly with a certain quantum state, one out of several which were previously in superposition. But this quantum state is not ***the*** quantum state. To acquire ***the*** quantum state, Bob must perform an operation on the precursor state himself – and the operation he must perform requires information from Alice to specify it. This key information is just ordinary, classical, information which must be transmitted in the normal way. So, not only has no matter or energy been transmitted faster than light, but nor does any information get transmitted transluminally by the teleportation process. Causality is respected as regards matter, energy *and* information.

2. Preliminaries – Quantum Gates

Before describing the teleportation process, it is necessary to appreciate what physical interactions Bob and Alice can perform on qubits in their possession. In the context of computation or communication the elementary interactions with qubits are referred to as 'gates', in analogy with classical computing. Also as with classical computing there is generally only one or two qubits which enter the gate, and one or two which are output from the gate.

The simplest example is the controlled-not gate, or CNOT. This takes two qubits as input and two qubits as output. Its operation is as follows: if the first qubit is in state $|0\rangle$ then the CNOT gate passes both qubits through to the output unchanged. But if the first qubit is $|1\rangle$, then the first qubit is unchanged but the second qubit is changed to the opposite state, i.e. $|0\rangle$ becomes $|1\rangle$ and vice-versa. Using a matrix notation in which the states are: 1 = $|0\rangle|0\rangle$; 2 = $|0\rangle|1\rangle$; 3 = $|1\rangle|0\rangle$; 4 = $|1\rangle|1\rangle$, CNOT can be represented by,

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \text{(QM10.1)}$$

CNOT is the only 2 state gate we shall need. But we also need some single qubit gates. Firstly there are the gates defined by the Pauli matrices,

$$\overline{\sigma} = \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] \qquad \text{(QM10.2)}$$

Hence, in the z-representation in which $|0\rangle \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ we have,

$$\sigma_z|0\rangle = |0\rangle;\ \sigma_z|1\rangle = -|1\rangle;\ \sigma_x|0\rangle = |1\rangle;\ \sigma_x|1\rangle = |0\rangle;\ \sigma_y|0\rangle = i|1\rangle;\ \sigma_y|1\rangle = -i|0\rangle \qquad \text{(QM10.3)}$$

Finally, we shall need the Hadamard gate which is defined by,

$$\hat{H}\text{ad} = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z) \qquad \text{(QM10.4)}$$

and hence, $\hat{H}\text{ad}|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $\hat{H}\text{ad}|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

All the above transformations are unitary and represent operations which can be realised physically.

3. The Quantum Teleportation Process

The simplest example is the teleportation of a single qubit state. Call the qubit to be teleported,

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \qquad \text{(QM10.5)}$$

Recall that neither Bob nor Alice know what state this is, i.e. they do not know α and β. The resource needed for teleportation to be possible is that Bob and Alice must share an entangled state of two other qubits. Call this two-qubit entangled state $|\phi\rangle$. For definiteness we specify it as,

$$|\phi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle + |1\rangle|1\rangle\right) \qquad \text{(QM10.6)}$$

The first of the qubits in $|\phi\rangle$ is accessible to Alice, whereas the second is accessible to Bob. For example, they may be a pair of spin ½ particles which have emerged from the decay of a spinless particle, as in the Bohm-Aharonov version of the EPR gedanken experiment. The initial state of all three qubits is thus,

$$\begin{aligned}|\psi\rangle|\phi\rangle &= \left(\alpha|0\rangle + \beta|1\rangle\right)\frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle + |1\rangle|1\rangle\right) \\ &= \frac{1}{\sqrt{2}}\left[\alpha|0\rangle|0\rangle|0\rangle + \alpha|0\rangle|1\rangle|1\rangle + \beta|1\rangle|0\rangle|0\rangle + \beta|1\rangle|1\rangle|1\rangle\right]\end{aligned} \qquad \text{(QM10.7)}$$

where the 1st and 2nd qubits are accessible to Alice, but only the 3rd qubit is accessible to Bob.

Alice now carries out some operations on the qubits she can influence. Firstly she carries out a CNOT, which has the following effect,

$$\text{CNOT}|\psi\rangle|\phi\rangle = \frac{1}{\sqrt{2}}\left[\alpha|0\rangle|0\rangle|0\rangle + \alpha|0\rangle|1\rangle|1\rangle + \beta|1\rangle|1\rangle|0\rangle + \beta|1\rangle|0\rangle|1\rangle\right] \qquad \text{(QM10.8)}$$

Next she carries out a Hadamard operation on the first qubit, the one that initially contained the state to be teleported. This gives,

$$\text{(QM10.9)}$$

$$\hat{\text{H}}\text{ad.CNOT}|\psi\rangle|\phi\rangle = \frac{1}{2}\left[\alpha\left(|0\rangle + |1\rangle\right)\left\{|0\rangle|0\rangle + |1\rangle|1\rangle\right\} + \beta\left(|0\rangle - |1\rangle\right)\left\{|1\rangle|0\rangle + |0\rangle|1\rangle\right\}\right]$$

This superposition of eight states can be re-arranged as follows,

$$\text{(QM10.10)}$$

$$\hat{\text{H}}\text{ad.CNOT}|\psi\rangle|\phi\rangle = \frac{1}{2}\left[|0\rangle|0\rangle\left(\alpha|0\rangle + \beta|1\rangle\right) + |0\rangle|1\rangle\left(\alpha|1\rangle + \beta|0\rangle\right) + |1\rangle|0\rangle\left(\alpha|0\rangle - \beta|1\rangle\right) + |1\rangle|1\rangle\left(\alpha|1\rangle - \beta|0\rangle\right)\right]$$

But the third qubit, the one accessible to Bob, can be written as $|\psi\rangle$, $\sigma_x|\psi\rangle$, $\sigma_z|\psi\rangle$ and $\sigma_x\sigma_z|\psi\rangle$ in the four terms respectively, i.e.,

$$\hat{\text{H}}\text{ad.CNOT}|\psi\rangle|\phi\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle|0\rangle + |0\rangle|1\rangle\sigma_x + |1\rangle|0\rangle\sigma_z + |1\rangle|1\rangle\sigma_x\sigma_z\right]|\psi\rangle \qquad \text{(QM10.11)}$$

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Let us review what the situation is now from the perspective of Alice. Her two qubits, the 1st and 2nd, occur with equal weight in the above superposition. Rather miraculously, the α and β parameters have switched from being associated with her qubits to being associated with Bob's, the 3rd qubit. From Bob's perspective, his qubit could be in any of the states $|\psi\rangle$, $\sigma_x|\psi\rangle$, $\sigma_z|\psi\rangle$ or $\sigma_x\sigma_z|\psi\rangle$. So, at this point, teleportation has not happened.

Alice now performs a measurement in the z-basis of the 2 x 2 Hilbert space accessible to her, i.e. she measures the 'spins' (if that's what they are) of her two qubits in the z-direction. The wavepacket collapses, i.e. the $\Re$-process causes just one of the four superposed states in (QM10.11) to be projected out. Alice obtains the result 00, or 01, or 10, or 11. Correspondingly the quantum state of Bob's qubit projects out to be either $|\psi\rangle$, or $\sigma_x|\psi\rangle$, or $\sigma_z|\psi\rangle$ or $\sigma_x\sigma_z|\psi\rangle$.

The important thing to realises is that, after Alice's measurement, Bob's quantum state is *deterministically* in one of the states $|\psi\rangle$, or $\sigma_x|\psi\rangle$, or $\sigma_z|\psi\rangle$ or $\sigma_x\sigma_z|\psi\rangle$. Prior to Alice's measurement, Bob's quantum state was a superposition of these states (with 'coefficients' entangled with the quantum states of Alice's qubits). This is crucial. The $\Re$-process which led to this deterministic state of affairs is the key step in the teleportation.

However, Bob does not yet possess the state $|\psi\rangle$. He would do so if only he knew which of the four states, $|\psi\rangle$, or $\sigma_x|\psi\rangle$, or $\sigma_z|\psi\rangle$ or $\sigma_x\sigma_z|\psi\rangle$, he currently had in hand. This is because each of the operators σ_x, σ_z and $\sigma_x\sigma_z$ is unitary and hence reversible. So, if Bob knew, say, that he currently had state $\sigma_x|\psi\rangle$, he would only need to carry out the physical operation corresponding to σ_x in order to recover the state $|\psi\rangle$, and hence to succeed in completing the teleportation.

But it is a simple matter for Bob to find out which state he has. He just needs to ask Alice. Her measurement has provided the result 00, or 01, or 10, or 11, and these correspond one-to-one with Bob's four quantum states $|\psi\rangle$, or $\sigma_x|\psi\rangle$, or $\sigma_z|\psi\rangle$ or $\sigma_x\sigma_z|\psi\rangle$. So, a simple classical communication of two bits of information provides Bob with the information he needs to carry out the final step of deconvolution to obtain the successfully teleported state, $|\psi\rangle$. Applause!

Note that Alice no longer has the state $|\psi\rangle$. She is left with one of the product states, $|0\rangle|0\rangle$ or $|0\rangle|1\rangle$ or $|1\rangle|0\rangle$ or $|1\rangle|1\rangle$ with equal probability. Hence, she is left with not the least vestige of information about her original quantum state $|\psi\rangle$, thus respecting the no-cloning theorem.

Also note that it has required the transmission of ***two*** bits of classical information to successfully teleport one qubit quantum state. In the next part of these Notes we shall see that the reverse of this also holds true. One qubit of quantum information suffices to transmit two bits of classical information.