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The Bree problem with the primary load cycling out-of-phase with the secondary load

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A R T I C L E I N F O

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A B S T R A C T

Ratchet and shakedown boundaries are derived analytically for the Bree problem with both primary and secondary loads cycling, but out of phase. Ratchet strains and cyclic plastic strains are derived in terms of the X,Y position on the ratchet diagram. The ratchet boundary lies between the Bree ratchet boundary and that for both loads cycling in-phase (or in exact anti-phase), but generally lies closer to the former. This provides a warning for ratcheting analyses with two loads cycling: employing the simplifying assumption of cycling exactly in-phase (or exactly in anti-phase) may be substantially non-conservative.

1. Introduction

In this, the 50th anniversary year of Bree's classic analysis of ratcheting [1], it is particularly appropriate to return to the algebraic method of solving ratcheting problems. Analytical solutions of ratcheting problems are relatively rare, being confined to cases of sufficient geometrical and loading simplicity. In real engineering applications, numerical methods, such as the linear matching method [2,3], are far more versatile and are now becoming mature techniques. Nevertheless, complete analytical solutions remain of considerable value, partly because they provide validation cases for numerical methods and partly because, in the few cases where they are tractable, they provide greater physical insight into the problem. This is exemplified by the Bree analysis, Refs. [1,4], which is still the chief landmark in this field even after half a century. A further advantage of a complete analytical solution is that all load magnitudes are addressed, identifying the behaviour at all points on the X,Y ratchet-shakedown diagram.

The Bree problem addresses uniaxial ratcheting under a constant primary membrane load plus a secondary wall-bending load which cycles between zero and some maximum value. Whilst the application that Bree had in mind in his original analysis was fast reactor fuel clad, and hence a cylindrical geometry, he analysed the problem as if for uniaxial stressing. Consequently the problem may be considered as relating to a beam of rectangular section.

A modified Bree problem consists of considering the primary membrane load to also cycle. If the primary and secondary loads cycle precisely in-phase (Fig. 1a), or if they cycle precisely in anti-phase (Fig. 1b), the complete solution to the resulting ratcheting and shakedown problem has been published, [5–7]. The in-phase and anti-phase cases have an identical ratchet boundary, but this lies at substantially greater loads than for the original Bree problem, i.e., in-phase or anti-phase cycling of the primary load is a significantly more benign loading as regards ratcheting than if the primary load were held constant.

However, whilst realistic plant operation does frequently involve both the primary and secondary loads cycling, the occurrence of precisely in-phase, or precisely anti-phase, cycling is improbable. This raises the question as to where the ratchet boundary lies when the primary and secondary loads cycle with some intermediate phase. There are published analyses in which the primary load cycles as well as the secondary load, with various phase relations, for example [8–10]. However, none have presented the complete solution for the out-of-phase case addressed here.

2. Definition of the problem addressed

The problem may be considered as relating to a beam of rectangular section subject to uniaxial stressing: a primary membrane load and a secondary bending load. Both loads vary between zero and some maximum, their temporal variation being 'square waves' with the same period but not in phase, as depicted by Fig. 2. Note that because a square-wave profile is assumed, each load is either acting fully or not acting at all. This simplifying assumption makes the problem analytically tractable.

The secondary bending stress is strain controlled and is

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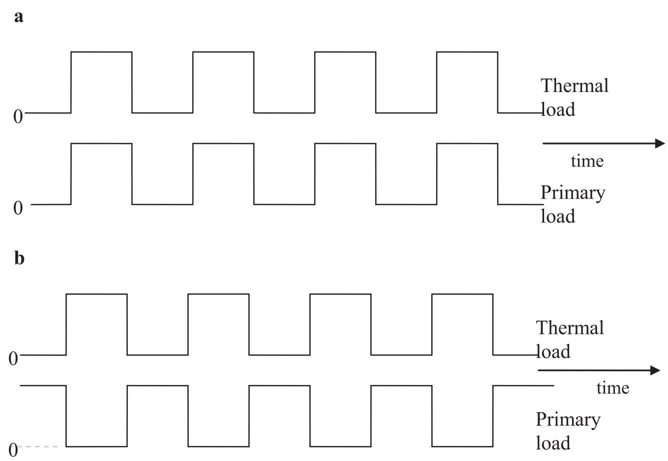

Fig. 1. (a) In-phase load sequence. (b) Anti-phase load sequence.

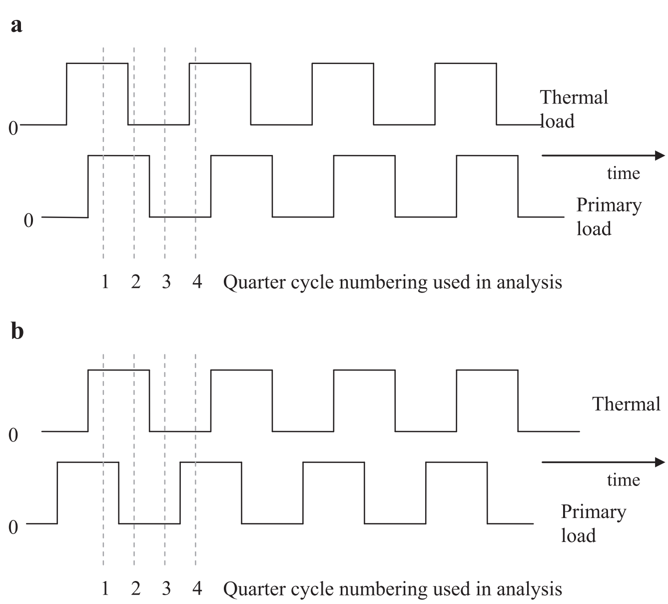

Fig. 2. (a) Positive-phase load sequence. (b) Negative-phase load sequence.

considered to be due to a uniform through-wall temperature gradient with the bending being fully restrained. Hence a net membrane strain might arise but the total bending strain is constrained to zero at all times.

In common with other analyses of the Bree type, elastic-perfectly plastic material behaviour is assumed.

Leaving aside the in-phase and anti-phase cases, there are two variants of the general phase problem. The loading sequence shown in Fig. 2a, which we will refer to as “positive phase”, is defined by,

[1] Thermal load plus primary load, then,
[2] Primary load alone, then,
[3] No load, then,
[4] Thermal load alone.

The cycle then repeats from Ref. [1]. Each of these load steps will be referred to as a “quarter cycle”. The whole cycle comprises the above sequence of four quarter cycles. The loading condition chosen as the starting point is irrelevant (once a reproducing sequence of stress distributions has been achieved). Only the order of the quarter cycles is relevant. The alternative loading sequence, as shown in Fig. 2b, is referred to as “negative phase” and is defined by,

[1] Thermal load plus primary load, then,
[2] Thermal load alone, then,
[3] No load, then,
[4] Primary load alone.

For example, if the thermal stress persists throughout a start-up transient, during which time the pressure increases from zero to full operating pressure, but ceases to act during steady operation at pressure, and does not recur during shutdown, then this would be the positive phase case. Conversely, if the thermal stress occurs in steady operation and persists during shutdown as the pressure falls to zero, but does not act when shut-down or during start-up to full pressure, then this would be the negative phase case.

In this work the complete ratchet-shakedown solution is presented for the positive phase case, thus illustrating the effect of out-of-phase loading when compared with the in-phase or anti-phase solution, [5–7]. Future work will address the negative phase case in order to illustrate the effect of load sequence on ratcheting.

3. Formulation of the problem and solution method

All dimensions are normalised by the section thickness, t. (This may differ from the normalisation convention used elsewhere, e.g. [7], normalised using $t/2$). All stresses are normalised by the yield stress, σ_y, and all strains by the yield strain, $\varepsilon_y = \sigma_y/E$. The applied (elastic) primary membrane stress, after normalisation, is X (hence $X = 1$ is plastic collapse). Similarly the peak (elastic) thermal bending stress after normalisation is Y. The through-thickness dimensionless coordinate x, is zero on the mid-section and $\pm 1/2$ on the surfaces. The stress, which varies with x position, is denoted σ. Equilibrium requires,

$$\text{If primary load is acting}: \int_{-1/2}^{+1/2} \sigma \cdot dx = X \tag{1}$$

$$\text{If primary load is not acting}: \int_{-1/2}^{+1/2} \sigma \cdot dx = 0 \tag{2}$$

The total strain ε, defined as the sum of the elastic, plastic and thermal strains, is required to be uniform across the section. It is given by,

$$\text{If the thermal load is acting}: \ \varepsilon = \sigma + \varepsilon_p - 2Yx \tag{3}$$

$$\text{If the thermal load is not acting}: \ \varepsilon = \sigma + \varepsilon_p \tag{4}$$

where ε_p is the plastic strain which varies with x. Equ.(3) assumes that the sign of the applied thermal stress is positive for $x > 0$. Because ε must be uniform through the section at all stages of the loading cycle, Equs.(3,4) have the following consequences, applicable at all points of the cross-section,

1) If the thermal load is acting, then,
   - ➢ Either, the slope of the ε_p versus x graph is zero and the slope of the σ versus x graph is $2Y$,
   - ➢ Or, the slope of the ε_p versus x graph is $2Y$ and the slope of the σ versus x graph is zero.
2) If the thermal load is not acting, then,

- ➢ Either, the slope of the ε_p versus x graph is zero and the slope of the σ versus x graph is zero,
- ➢ Or, the slope of the ε_p versus x graph is $2Y$ and the slope of the σ versus x graph is $-2Y$.

3) If $|\sigma| < \sigma_y$ on quarter cycle n then ε_p is unchanged from its value on quarter cycle $n-1$.

The above rules allow the stress and plastic strain distributions to be found uniquely for a small number of discrete cases which together address all possible (X, Y) values. The problem is reduced to graphical construction of stress distributions and plastic strain distributions which obey these rules. The main labour consists of an exhaustive search for all such distributions, and the identification of the region on the (X, Y) ratchet diagram to which each applies.

4. Ratchet solutions

Figs. 3a–10a are schematic diagrams of possible stress distributions. Each Figure shows the stress distributions for the first five quarter-cycles. The stress distribution for the fifth quarter cycle differs from that of the first quarter cycle despite corresponding to the same loading. Thereafter the distributions repeat, i.e., the sixth cycle has the same stress distribution as the second cycle, the seventh cycle has the same stress distribution as the third cycle, etc.

Figs. 3b–10b are the plastic strain distributions which are compatible with the stress distributions of Figs. 3a–10a according to the rules of §3.

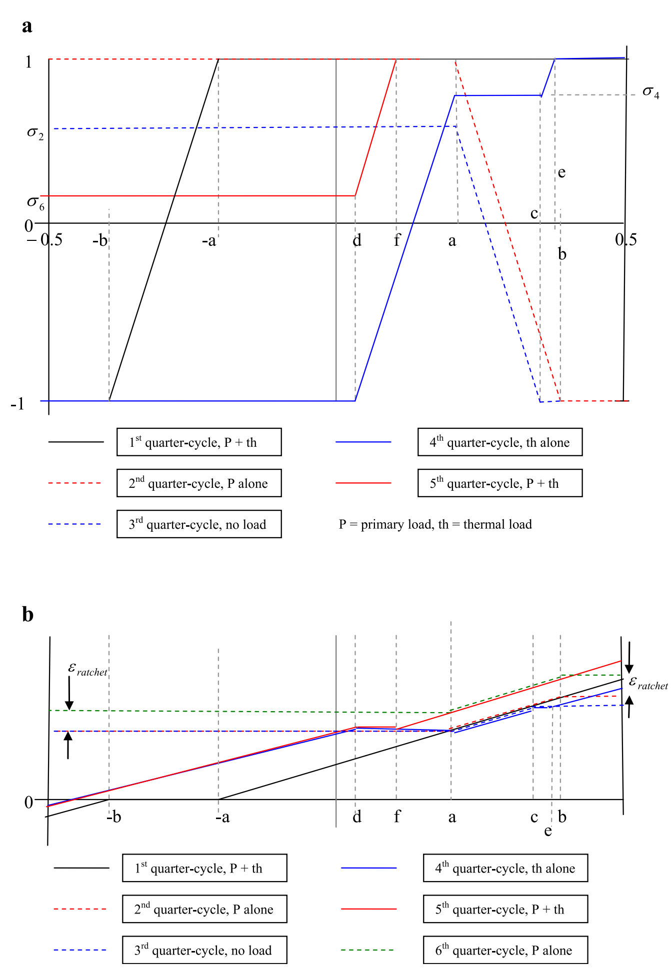

Fig. 3. (a) Stress Distributions leading to Ratchet Region R2a on Fig. 15. (b) Plastic Strain Distributions Corresponding to Fig. 3a Stresses.

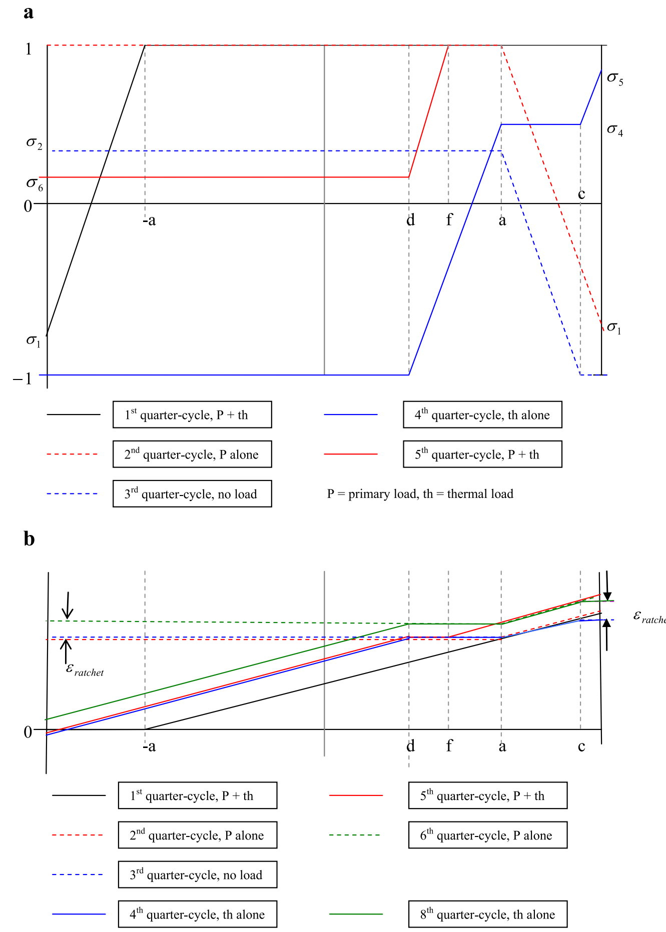

Fig. 4. (a) Stress Distributions leading to Ratchet Region R1c on Fig. 15. (a)Plastic Strain Distributions Corresponding to Fig. 4a Stresses.

All of Figs. 3–10 are ratcheting conditions. This is clear from the fact that tensile yielding occurs at all points of the cross-section at some time in the loading cycle. The strain plots identify the ratchet strains. There are thus eight qualitatively distinct ratcheting regions for the out-of-phase problem (as compared with just two such regions for the original Bree problem [1], or the in-phase or anti-phase cases [5–7]).

Algebraic expressions for the dimensions a to f and the stresses σ_1 to σ_6 and σ_{10} are derived for each of Figs. 3–10 (where applicable). These quantities are uniquely determined in terms of X and Y. It will be shown that Figs. 3, 4 and 10 define the lower ratchet boundary for different ranges of X, whilst Figs. 5–9 apply at higher Y values, providing a sub-division of the overall ratcheting region. It was necessary to analyse all eight qualitatively different types of stress distributions because, until this was done, it was not obvious which would turn out to be the most limiting, i.e., to correspond to the lowest ratchet boundary on the X, Y diagram. Moreover, to calculate the ratchet strain at any given X, Y position, it is necessary to identify correctly the relevant regime, i.e., which of Figs. 3–10 applies.

The following sub-sections describe how the eight ratcheting regions, shown in Fig. 15, are identified. In each case this is done by applying, for each quarter cycle, the rules in §3 for the slopes of the stress distributions and the constraint of equilibrium given by Equs.(1) and (2). By this means the stress and position parameters of Figs. 3–10 may be derived and are given in Tables 1–8.

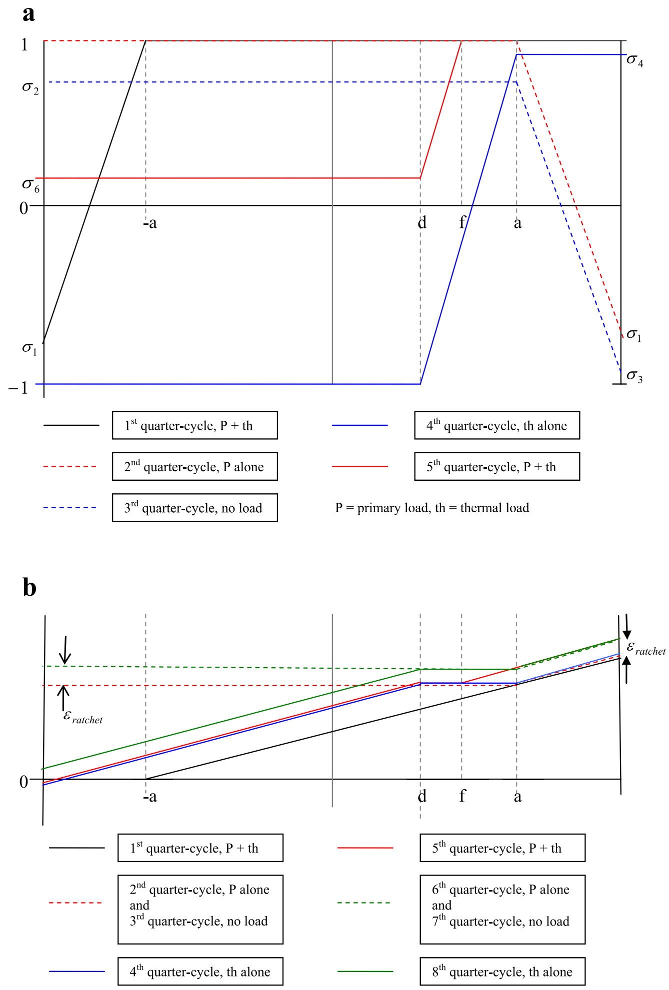

Fig. 5. (a)Stress Distributions leading to Ratchet Region R1a on Fig. 15. (b) Plastic Strain Distributions Corresponding to Fig. 5a Stresses.

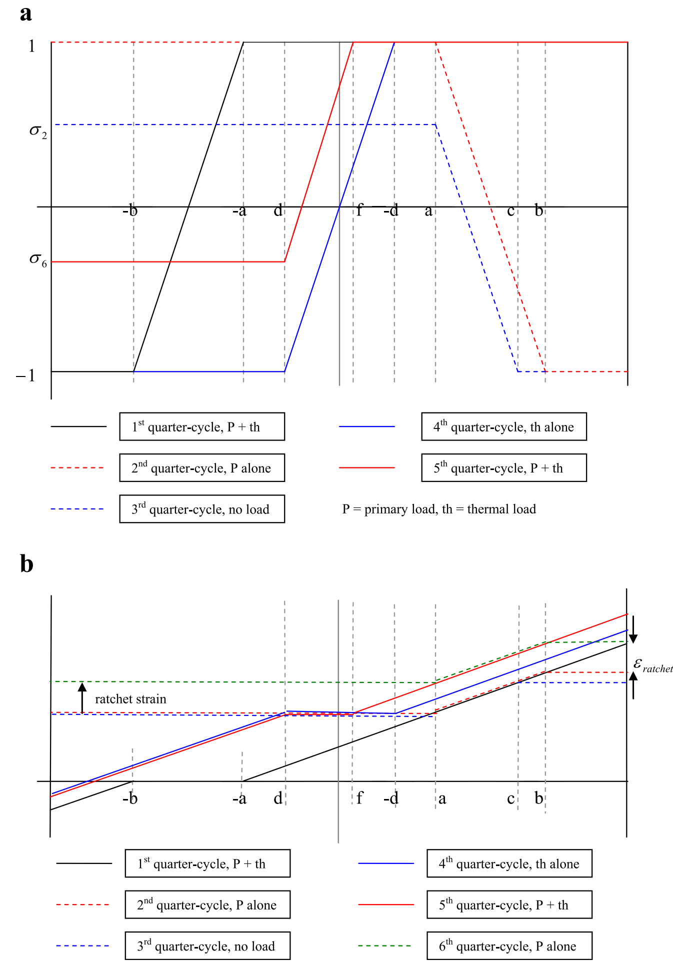

Fig. 6. (a) Stress Distributions leading to Ratchet Region R2b on Fig. 15. (b) Plastic Strain Distributions Corresponding to Fig. 6a Stresses.

4.1. *Distributions of Fig. 3*

If Fig. 3a applies then tensile yielding occurs at all points of the cross section at some time during the loading cycle provided that $f < a$. This is the condition for ratcheting and Fig. 3b gives the ratchet strain to be $\varepsilon_{ratchet} = 2Y(a - f)$.

The ratchet boundary is given by $f = a$ which defines a curve on the X, Y diagram which may be solved numerically using the solutions for the parameters given in Table 1. This curve is valid as part of the ratchet boundary only so long as the dimensions a, b, c, d, e, f all lie in the physical range -0.5 to $+0.5$, and the stresses $\sigma_2, \sigma_4, \sigma_6$ lie in the physical range -1 to $+1$. The numerical evaluation for increasing X shows that these conditions are first violated when b and e reach 0.5 at $X = 0.5127$. Fig. 3 thus defines the ratchet boundary denoted GA in Fig. 15, applying in the range $0 < X \le 0.5127$. The lower bound of this X range corresponds to $Y \to \infty$ whilst the upper bound corresponds to $Y = 2.0521$, labelled point A on Fig. 15.

Note that, from the above solution, $b = 0.5$ when $Y(1 - X) = 1$, so the point $(0.5127, 2.0521)$ necessarily lies on this hyperbola. More generally, for Fig. 3 to be valid it is required that X, Y should lie above the hyperbola $Y(1 - X) = 1$, shown as ACK in Fig. 15.

Fig. 3 reduces to Fig. 6 when $\sigma_4 \to 1$, or equivalently $c \to e$, which is easily shown using the solutions of Table 1 to reduce to the condition $XY = 2$. Fig. 3 applies below the hyperbola $XY = 2$, shown as CH on Fig. 15, ensuring that $\sigma_4 < 1$. In summary, Fig. 3 applies in the region of the X, Y diagram which lies,

- Above the Fig. 3 ratchet boundary, $f = a$ (curve GA on Fig. 15);
- Above the hyperbola $Y(1 - X) = 1$ (curve ACK on Fig. 15);
- Below the hyperbola $XY = 2$ (curve CH on Fig. 15).

This region is denoted R2a on Fig. 15. That the region immediately below R2a (denoted P2 on Fig. 15) does not ratchet will be shown below (§5.4).

Note that, for the case of in-phase or anti-phase cycling, the hyperbola $XY = 2$ is part of the ratchet boundary (see Refs. [5–7]). Consequently region R2a lies *below* the ratchet boundary for in-phase or anti-phase cycling, indicating that the out-of-phase case ratchets more readily.

4.2. *Distributions of Fig. 4*

The ratchet boundary is given by $f = a$ which defines a curve on the X, Y diagram which may be solved numerically using the parameter solutions of Table 2. This curve is valid as a part of the ratchet boundary only so long as the dimensions a, c, d, f all lie in the physical range -0.5 to $+0.5$, and the stresses $\sigma_1, \sigma_2, \sigma_4, \sigma_5, \sigma_6$ all lie in the physical range -1 to $+1$. The numerical evaluation shows that these conditions are met on the ratchet boundary only for X in the range $0.5127 \le X \le 0.6667$. Hence, over this range of X the

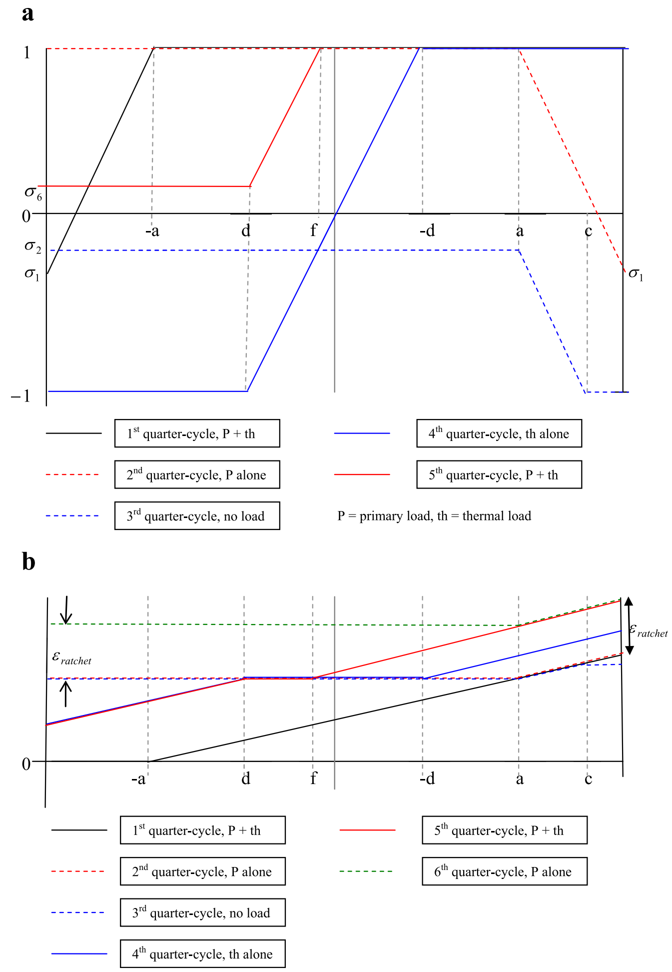

Fig. 7. (a) Stress Distributions leading to Ratchet Region R1e on Fig. 15. (b) Plastic Strain Distributions Corresponding to Fig. 7a Stresses.

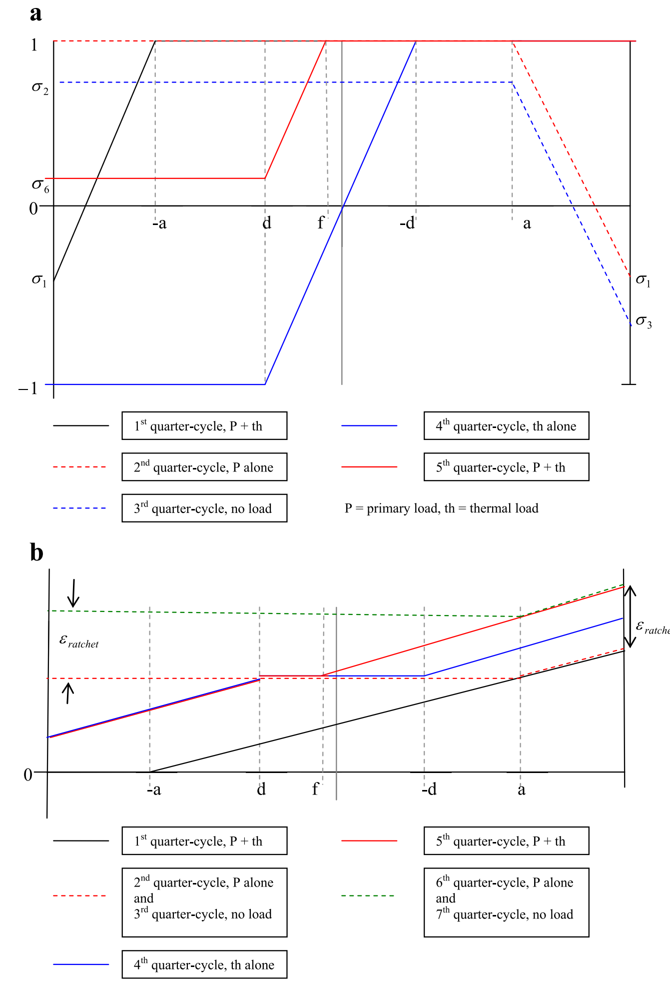

Fig. 8. (a) Stress Distributions leading to Ratchet Region R1b on Fig. 15. (b) Plastic Strain Distributions Corresponding to Fig.8a Stresses.

boundary of the ratcheting region defined by Fig. 4 is the curve AB on Fig. 15, the end points of which have $Y = 2.0521$ and $Y = 1.3333$ respectively. It will be shown later (§5.3) that the region immediately below AB does not ratchet, so AB is part of the overall ratchet boundary. The ratchet region immediately above AB is denoted region R1c on Fig. 15.

At the lower extremity of the Fig. 4 ratchet boundary, $X, Y = 0.6667,\ 1.3333$, denoted B on Fig. 15, $c = 0.5$, indicating that a larger X would not be consistent with compressive yielding occurring on the third quarter cycle, so that Fig. 4 then morphs into Fig. 5. The condition $c = 0.5$ therefore provides the boundary between R1c and the neighbouring region, denoted R1a on Fig. 15, namely the curve UBDL given by $Y = \frac{(2-X)^2}{4(1-X)}$.

Finally, there is an upper boundary to the region consistent with Fig. 4 defined by the requirement that $\sigma_5 < 1$ which identifies the region R1c to be bounded by curve AD on Fig. 15.

4.3. *Distributions of Fig. 5*

For Fig. 5a to be valid it is required that $\sigma_3 > -1$ and $\sigma_4 < 1$ and $d > -0.5$. Using the parameter solutions of Table 3, these conditions are found to correspond to regions on the X, Y diagram as follows,

$$\sigma_3 > -1 \Rightarrow Y < \frac{(2-X)^2}{4(1-X)} \tag{5}$$

$$\sigma_4 < 1 \Rightarrow Y < \left(\sqrt{2-X} + \sqrt{1-X}\right)^2 \tag{6}$$

$$d > -0.5 \Rightarrow Y > 2 - X \tag{7}$$

Equ.(5) is recognised as curve UBDL on Fig. 15, whilst Equ.7 is shown as the line JBF on Fig. 15, so Fig. 5 corresponds to a region above BF but below (to the right of) BDL. Finally, Equ. (6) is plotted on Fig. 15 as the curve CDF, so that DF establishes the upper boundary of the region to which Fig. 5 applies, denoted R1a on Fig. 15.

Ratcheting requires $f < a$, and this condition may be simplified and shown to be,

$$Y > \left(1 - \frac{X}{2}\right)^2 \left[\sqrt{2-X} + \sqrt{1-X}\right]^2 \tag{8}$$

But to the right of point B in Fig. 15 ($X > 2/3$), the curve given by Equ. (8) lies below the curve BF. This means that the ratchet strain is non-zero on the lower boundary of region R1a, i.e., on BF, which implies that there is a further ratchet region below it. (It is shown below, §4.8, that this region, R1f, relates to Fig. 10).

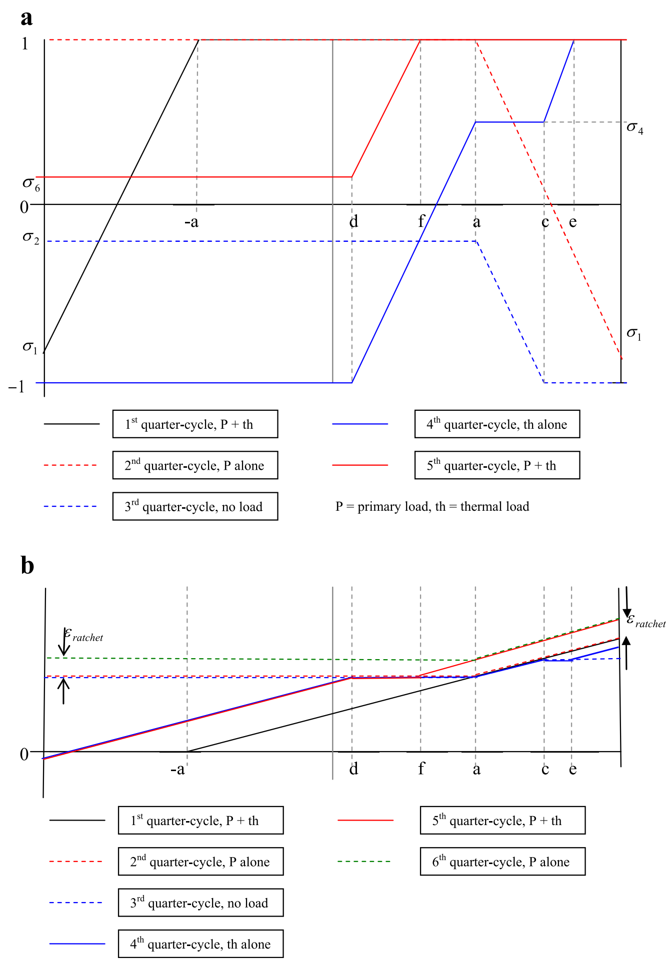

Fig. 9. (a) Stress Distributions leading to Ratchet Region R1d on Fig. 15. (b) Plastic Strain Distributions Corresponding to Fig.9a Stresses.

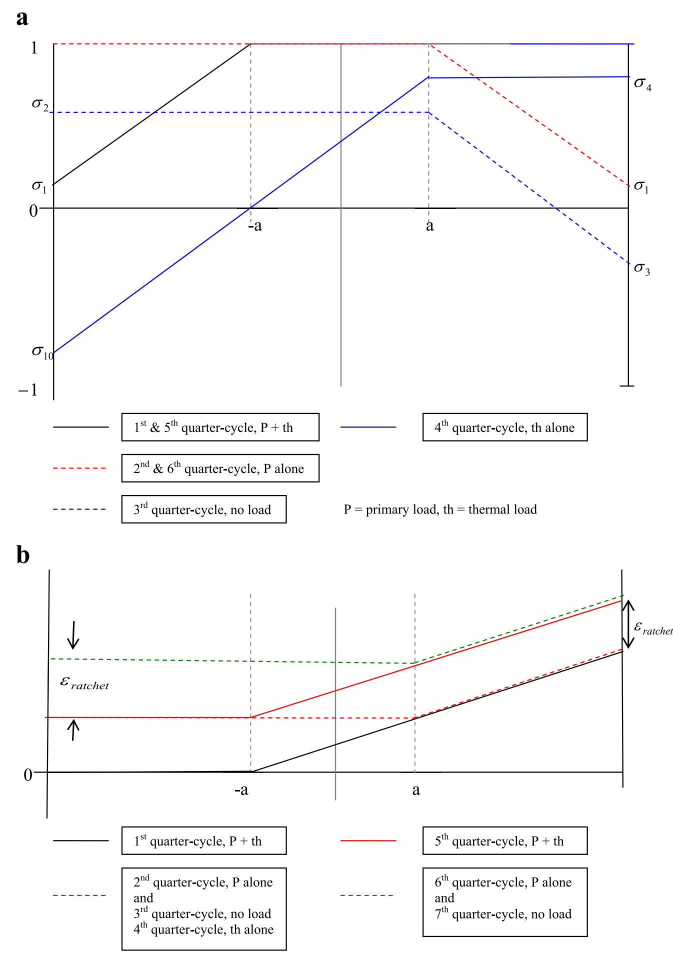

Fig. 10. (a) Stress Distributions leading to Ratchet Region R1f on Fig. 15. (b) Plastic Strain Distributions Corresponding to Fig.10a Stresses.

Note that Equ.(6) can also be written $1-X>\frac{1}{4}(Y-1)\left(1-\frac{1}{Y}\right)$ and hence curve CDF.

Coincides with the lower boundary of the R1 ratchet region for the in-phase/anti-phase solution presented in Refs. [5–7]. Thus the ratchet regions R1a, R1c, R1d and R1f all lie below the ratchet boundary for perfect in-phase/anti-phase loading, demonstrating again, as we found for region R2a, that out-of-phase loading more readily gives rise to ratcheting than the in-phase or perfectly anti-phase cases.

4.4. Distributions of Fig. 6

Fig. 6a applies if $b<0.5$ which, using the parameter solutions of Table 4, requires $Y>1/(1-X)$, i.e., above curve ACK on Fig. 15. Fig. 6a becomes the same as Fig. 3a when $-d=a$ on Fig. 6a (equivalently $\sigma_4=1$ on Fig. 3a). This requires $Y>2/X$, i.e., above the curve HC on Fig. 15. These two conditions define region R2b on Fig. 15. Since this region lies above R2a it can be anticipated to be a ratcheting region and this is confirmed by noting that the ratchet strain $\varepsilon_{ratchet}=2Y(a-f)$, is non-zero provided $f<a$ and that the latter condition reduces to,

$$Y>\frac{2(2-X)}{X(2+X)} \tag{9}$$

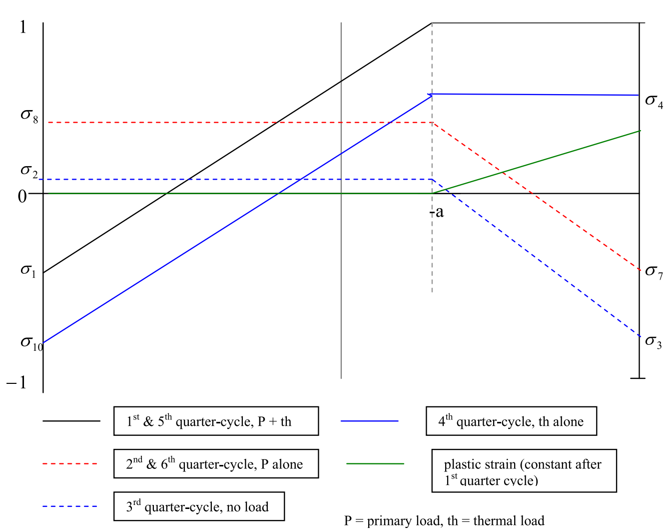

Fig. 11. Stress and Strain Distributions leading to Shakedown Region S1.

and this condition is obviously fulfilled in region R2b by virtue of lying above HC, i.e. because $Y>2/X$. Region R2b is identical to the ratcheting region R2 for in-phase or anti-phase loading, [5–7].

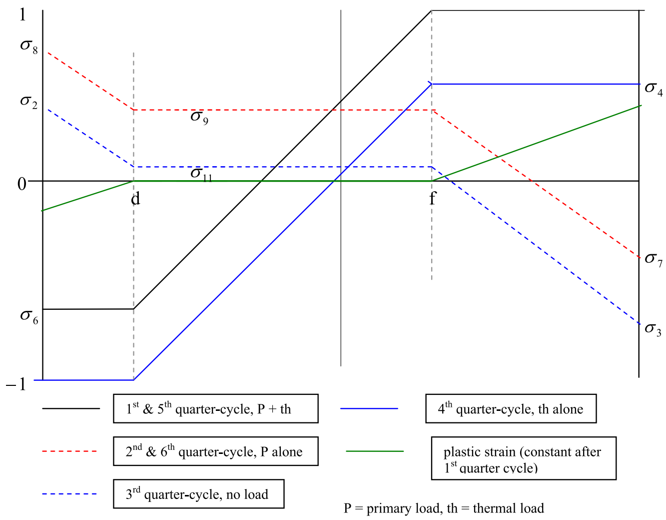

Fig. 12. Stress and Strain Distributions leading to Shakedown Region S2.

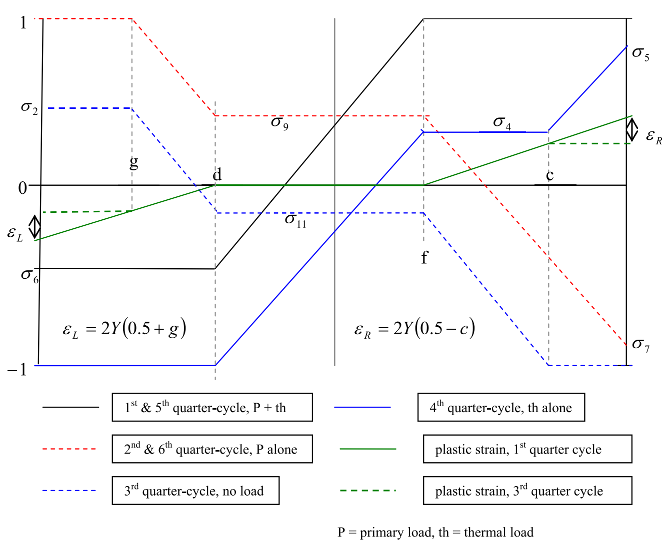

Fig. 13. Stress and Strain Distributions leading to Plastic Cycling Region P1.

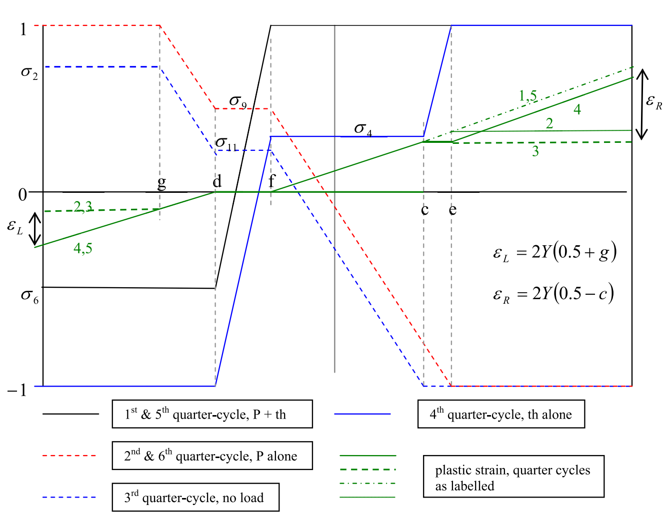

Fig. 14. Stress and Strain Distributions leading to Plastic Cycling Region P2.

4.5. Distributions for Fig. 7

Using the parameter solutions in Table 5, the requirement $\sigma_1 > -1$ leads to $Y(1-X) < 1$, identifying the region in question as lying below (to the right of) curve ACK on Fig. 15.

The requirement $c < 0.5$ leads to $Y > \frac{(2-X)^2}{4(1-X)}$, identifying the region in question as lying above (to the left of) curve UBDL on Fig. 15.

Finally, it is also necessary that $-d < a$ because, if $-d > a$ it would not be possible to fulfill the requirements of §3 for the plastic strain distributions in the X range $[a, -d]$. The requirement $-d < a$ reduces to $Y > (\sqrt{2-X} + \sqrt{1-X})^2$, which may also be written $1 - X < \frac{1}{4}(Y-1)\left(1 - \frac{1}{Y}\right)$, i.e., the region to which Fig. 7 applies lies above curve CDF on Fig. 15.

These three conditions identify Fig. 7 as corresponding to the region denoted R1e on Fig. 15.

4.6. Distributions of Fig. 8

For Fig. 8 to be valid it is required that $\sigma_3 > -1$ which gives, using the parameter solutions in Table 6, $Y < \frac{(2-X)^2}{4(1-X)}$, and so the region in question is identified as lying below (to the right of) UBDL in Fig. 15.

For the plastic strains of Fig. 8b to be consistent with the rules of §3 it is also required that $-d < a$ and this reduces to $Y > (\sqrt{2-X} + \sqrt{1-X})^2$, so the region in question lies above CDF in Fig. 15.

Hence Fig. 8 corresponds to the region denoted R1b on Fig. 15.

Note that the union of regions R1b and R1e equals the ratcheting region R1 for in-phase or anti-phase cycling, [5–7].

4.7. Distributions of Fig. 9

So far regions R1a, R1b, R1c, R1e, R2a and R2b on Fig. 15 have been constructed. Clearly there must be a ratcheting region as designated R1d on Fig. 15. Here it is shown that Fig. 9 corresponds to region R1d.

Firstly, using the parameter solutions of Table 7, the requirement $\sigma_1 > -1$ gives $Y(1-X) < 1$, establishing that the region lies below (to the right of) curve ACK.

Secondly, when $e \to 0.5$ Fig. 9a becomes the same as Fig. 4a when $\sigma_5 \to 1$, and the latter defines the curve AD. This establishes that the Fig. 9 region is also bounded by AD.

Finally, in the limit $d \to -a$ Fig. 9a morphs into Fig. 7a, hence the lower boundary of the latter is an upper boundary for Fig. 9 and this has already been shown to be given by $Y = (\sqrt{2-X} + \sqrt{1-X})^2$, i.e., curve CD on Fig. 15.

Hence Fig. 9 corresponds to the region denoted R1d on Fig. 15, as expected.

4.8. Distributions of Fig. 10

In §4.3 it was noted that the lower boundary of region R1a, BF on Fig. 15, had non-zero ratchet strain, and hence that there must be another ratchet region lying below it. Here it is shown that this region, denoted R1f on Fig. 15, corresponds to the stress and plastic strain distributions of Fig. 10.

The parameters of Fig. 10 are found in the usual way and are given in Table 8. For Fig. 10a to be valid it is required that $\sigma_{10} > -1$ which implies $Y < 2 - X$ and hence the region in question lies below BF on Fig. 15. Also, for Fig. 10a to be a ratcheting condition it is required that $a > 0$, because the ratchet strain is given by $4aY$. This implies $Y > 4(1-X)$ and hence the region in question lies above BE on Fig. 15. In conclusion, Fig. 10 corresponds to region R1f on Fig. 15,

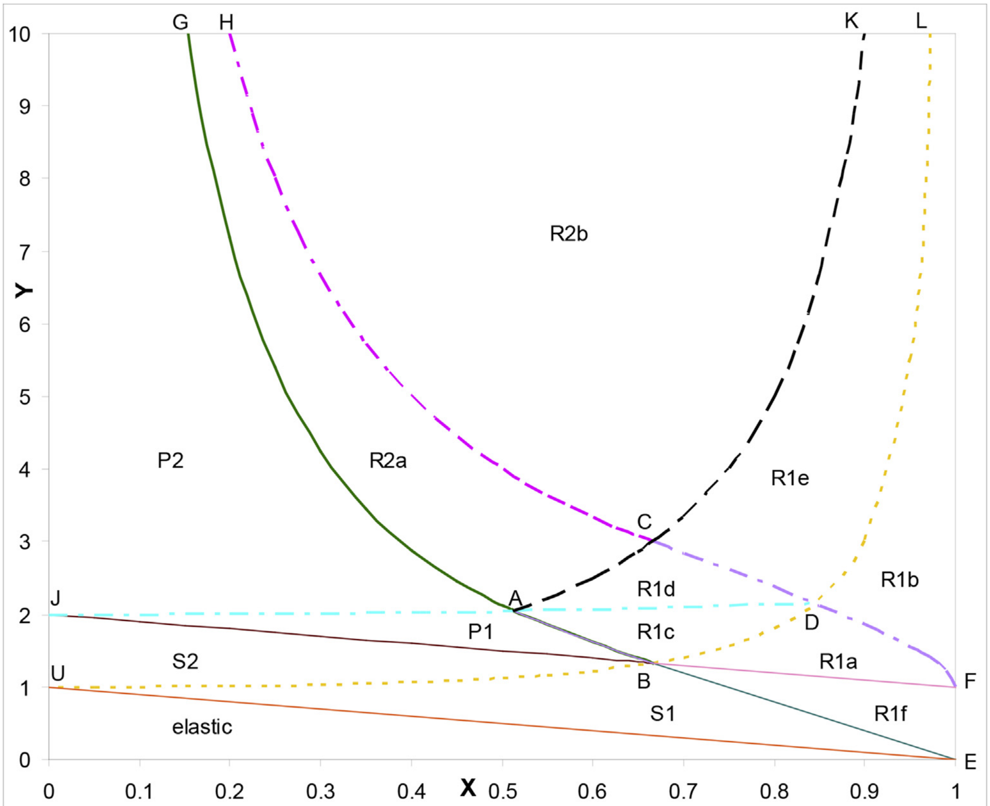

Fig. 15. Ratchet-shakedown diagram for positive out-of-phase cycling.

Table 1
Parameters for Fig. 3.

$2a = X - \frac{1}{Y}$	$2b = X + \frac{1}{Y}$	$c = \left[\left(a+\frac{1}{2}\right)^2 + \frac{1}{Y}\right]^{1/2} - \frac{1}{2}$
$d = a - \frac{c+\frac{1}{2Y}}{Y(c-a)+1}$	$e = \frac{1}{Y} - a + c + d$	$f = \left[\left(d+\frac{1}{2}\right)^2 + \frac{1-X}{Y}\right]^{1/2} - \frac{1}{2}$
$\varepsilon_{ratchet} = 2Y(a-f)$	$\sigma_2 = Y(2c - X)$	$\sigma_4 = 2Y(a-d) - 1$
$\sigma_6 = 1 - 2Y(f-d)$		

Table 2
Parameters for Fig. 4.

$a = \frac{1}{2} - \sqrt{\frac{1-X}{Y}}$	$c = \left[\left(a+\frac{1}{2}\right)^2 + \frac{1}{Y}\right]^{1/2} - \frac{1}{2}$
$d = \frac{1}{2} - \sqrt{\frac{1}{Y} + \left(\frac{1}{2}-a\right)^2 - \left(\frac{1}{2}-c\right)^2}$	$f = \left[\left(d+\frac{1}{2}\right)^2 + \frac{1-X}{Y}\right]^{1/2} - \frac{1}{2}$
$\varepsilon_{ratchet} = 2Y(a-f)$	$\sigma_1 = 1 - 2\sqrt{Y(1-X)}$
$\sigma_2 = 2Y(c-a) - 1$	$\sigma_4 = 2Y(a-d) - 1$
$\sigma_5 = Y(1+2a-2c-2d) - 1$	$\sigma_6 = 1 - 2Y(f-d)$

Table 3
Parameters for Fig. 5.

$a = \frac{1}{2} - \sqrt{\frac{1-X}{Y}}$	$d = \frac{1}{2} - \sqrt{\frac{2-X}{Y}}$
$f = \left[\left(\frac{1}{2}+d\right)^2 + \frac{1-X}{Y}\right]^{1/2} - \frac{1}{2}$	$\varepsilon_{ratchet} = 2Y(a-f)$
$\sigma_1 = 1 - 2\sqrt{Y(1-X)}$	$\sigma_2 = 1 - X$
$\sigma_3 = 1 - X - 2\sqrt{Y(1-X)}$	$\sigma_4 = 2Y(a-d) - 1$
$\sigma_6 = 1 - 2Y(f-d)$	

Table 4
Parameters for Fig. 6.

$2a = X - \frac{1}{Y}$	$2b = X + \frac{1}{Y}$	$c = \left[\left(a+\frac{1}{2}\right)^2 + \frac{1}{Y}\right]^{1/2} - \frac{1}{2}$
$d = -\frac{1}{2Y}$	$f = \left[\left(d+\frac{1}{2}\right)^2 + \frac{1-X}{Y}\right]^{1/2} - \frac{1}{2}$	$\varepsilon_{ratchet} = 2Y(a-f)$
$\sigma_2 = Y(2c - X)$	$\sigma_6 = 1 - 2Y(f-d)$	

Table 5
Parameters for Fig. 7.

$a = \frac{1}{2} - \sqrt{\frac{1-X}{Y}}$	$c = \left[\left(a+\frac{1}{2}\right)^2 + \frac{1}{Y}\right]^{1/2} - \frac{1}{2}$
$d = -\frac{1}{2Y}$	$f = \left[\left(\frac{1}{2}+d\right)^2 + \frac{1-X}{Y}\right]^{1/2} - \frac{1}{2}$
$\varepsilon_{ratchet} = 2Y(a-f)$	$\sigma_1 = 1 - 2\sqrt{Y(1-X)}$
$\sigma_2 = 2Y(c-a) - 1$	$\sigma_6 = 1 - 2Y(f-d)$

Table 6
Parameters for Fig. 8.

$a = \frac{1}{2} - \sqrt{\frac{1-X}{Y}}$	$d = -\frac{1}{2Y}$
$f = \left[\left(\frac{1}{2}+d\right)^2 + \frac{1-X}{Y}\right]^{1/2} - \frac{1}{2}$	$\varepsilon_{ratchet} = 2Y(a-f)$
$\sigma_1 = 1 - 2\sqrt{Y(1-X)}$	$\sigma_2 = 1 - X$
$\sigma_3 = 1 - X - 2\sqrt{Y(1-X)}$	$\sigma_6 = 1 - 2Y(f-d)$

and the lower boundary of this region, line BE, has zero ratchet strain and hence forms part of the overall ratchet boundary.

Table 7
Parameters for Fig. 9.

$a = \frac{1}{2} - \sqrt{\frac{1-X}{Y}}$	$c = \left[\left(a+\frac{1}{2}\right)^2 + \frac{1}{Y}\right]^{1/2} - \frac{1}{2}$
$d = a - \frac{c+\frac{1}{2Y}}{Y(c-a)+1}$	$e = \frac{1}{Y} - a + c + d$
$f = \left[\left(\frac{1}{2}+d\right)^2 + \frac{1-X}{Y}\right]^{1/2} - \frac{1}{2}$	$\varepsilon_{ratchet} = 2Y(a-f)$
$\sigma_1 = 1 - 2\sqrt{Y(1-X)}$	$\sigma_2 = 2Y(c-a) - 1$
$\sigma_4 = 2Y(a-d) - 1$	$\sigma_6 = 1 - 2Y(f-d)$

Table 8
Parameters for Fig. 10.

$a = \frac{1}{2} - \sqrt{\frac{1-X}{Y}}$	$\varepsilon_{ratchet} = 4aY$
$\sigma_1 = 1 - 2\sqrt{Y(1-X)}$	$\sigma_2 = 1 - X$
$\sigma_3 = 1 - X - 2\sqrt{Y(1-X)}$	$\sigma_4 = 1 - X + Y - 2\sqrt{Y(1-X)}$
$\sigma_{10} = 1 - X - Y$	

5. Shakedown and cyclic plasticity regions

5.1. Distributions of Fig. 11

For Fig. 11 to be valid it is required that $-a < 0.5$ which, using the parameter solutions of Table 9, implies $Y > 1 - X$. This is merely the condition that yielding does occur when both loads act, and establishes that the region in question lies above the 'elastic line', UE in Fig. 15.

For Fig. 11 to be valid it is also required that $\sigma_8 < 1$ which implies that $Y < 4(1-X)$, i.e., that the region in question lies below line BE on Fig. 15.

Finally, for Fig. 11 to be valid it is also required that $\sigma_9 > -1$ which implies that $Y < \frac{(2-X)^2}{4(1-X)}$, i.e., that the region in question lies below curve UBDL on Fig. 15.

These conditions establish that Fig. 11 corresponds to region S1 on Fig. 15. Referring to Fig. 11b, there is no plastic deformation after the first quarter cycle, so this region indicates strict shakedown to elastic cycling.

5.2. Distributions of Fig. 12

The region adjoining region S1 above curve UB is also a shakedown region (S2) corresponding to the distributions shown in Fig. 12. This is demonstrated here. The parameters for Fig. 12 are found as before and are listed in Table 10.

The requirement that $\sigma_8 < 1$, or equivalently that $\sigma_3 > -1$, lead to $Y < 2 - X$ and hence that the region in question lies below line JBF in Fig. 15. The requirement that $d < 0.5$ implies that $Y > \frac{(2-X)^2}{4(1-X)}$ and hence that the region in question lies above curve UBDL in Fig. 15. So Fig. 12 corresponds to the region S2 on Fig. 15, as claimed. Referring to Fig. 12b, there is no plastic straining after the first quarter cycle, only elastic cycling thereafter, so S2 is a strict shakedown region.

Table 9
Parameters for Fig. 11.

$a = \frac{1}{2} - \sqrt{\frac{1-X}{Y}}$	$\sigma_1 = 1 - 2\sqrt{Y(1-X)}$
$\sigma_2 = 1 - X + Y - 2\sqrt{Y(1-X)}$	$\sigma_3 = 1 - X - Y$
$\sigma_4 = 1 - X$	$\sigma_7 = 1 - Y$
$\sigma_8 = 1 + Y - 2\sqrt{Y(1-X)}$	$\sigma_{10} = 1 - X - 2\sqrt{Y(1-X)}$

Table 10
Parameters for Fig. 12.

$2d = -\left[\frac{2-X}{2Y} + \frac{X}{2-X}\right]$	$2f = \left[\frac{2-X}{2Y} - \frac{X}{2-X}\right]$
$\sigma_{11} = -\frac{X}{2} + \frac{XY}{2-X}$	$\sigma_2 = Y - 1$
$\sigma_3 = 1 - X - Y$	$\sigma_4 = 1 - X$
$\sigma_6 = -\sigma_4$	$\sigma_7 = 1 - Y$
$\sigma_8 = X + Y - 1$	$\sigma_9 = \frac{X}{2} + \frac{XY}{2-X}$

5.3. Distributions of Fig. 13

The only regions on Fig. 15 which remain to be classified are those denoted P1 and P2. Here we show that the stress and strain distributions of Fig. 13 correspond to region P1, and also that this is a region which neither ratchets nor displays shakedown to elastic cycling. Instead it exhibits plastic cycling on both surfaces.

Using the rules of §3 as before the parameters of Fig. 13 are found, in this case by numerical solution of the resulting simultaneous equations. From Fig. 13b we see that the cyclic plastic strain range on the left surface is $\varepsilon_L = 2Y(0.5 + g)$ whilst the cyclic plastic strain range on the right surface is $\varepsilon_R = 2Y(0.5 - c)$. Consequently, the region in question must give way to elastic cycling (shakedown) when $g = -0.5$ and $c = 0.5$. Both these conditions are fulfilled on the line JB, i.e., $Y = 2 - X$, thus permitting the identification of Fig. 13 with the region immediately above S2.

For Fig. 13 to be valid it is also required that $\sigma_5 < 1$ and $\sigma_7 > -1$. In the range $0 < X < 0.5127$ both these limits produce the boundary shown as JA on Fig. 15, which thus marks the upper limit of the region in which Fig. 13 applies to the left of point A. To the right of point A, and as far as point B, the requirement $\sigma_9 < 1$ is limiting and produces the upper boundary denoted AB in Fig. 15. Hence, Fig. 13 corresponds to region P1 on Fig. 15.

5.4. Distributions of Fig. 14

The only region on Fig. 15 now left is P2. Here we show that it corresponds to the stress and plastic strain distributions of Fig. 14. Using the rules of §3 as before the parameters of Fig. 14 are found, in this case by numerical solution of the resulting simultaneous equations. The lower boundary of the region in which Fig. 14 applies is given by $e \to 0.5$, when Fig. 14 morphs into Fig. 13. Numerical evaluation confirms that this reproduces curve JA on Fig. 15. The upper limit of the region in which Fig. 14 applies is defined by $\sigma_9 \to 1$, which clearly marks a boundary with a ratcheting condition. Again numerical evaluation confirms that the boundary in question coincides with curve GA on Fig. 15. It is concluded that Fig. 14 corresponds to region P2 on Fig. 15, lying above region P1 but immediately below the ratcheting region R2a.

6. Ratchet-shakedown diagram

The resulting ratchet-shakedown diagram is shown, with all its sub-regions, in Fig. 15. Coordinates of salient points on Fig. 15 are given in Table 11a. Table 11b gives the algebraic equations of the

Table 11a
Significant points on the ratchet diagram (Fig. 15).

Point	X	Y
A	0.5127	2.0521
B	0.6667	1.3333
C	0.6667	3
D	0.8453	2.1547
F	1	1

Table 11b
Boundary curves on the ratchet diagram (Fig. 15): Algebraic.

Curve or Line	Equation
ACK	$Y(1-X)=1$
HC	$XY=2$
UBDL	$Y=\frac{(2-X)^2}{4(1-X)}$
JBF	$Y=2-X$
UE	$Y=1-X$
CDF	$Y=(\sqrt{2-X}+\sqrt{1-X})^2$
BE	$Y=4(1-X)$

various boundary curves on Fig. 15, except in two cases, curves GAB and JAD, which are given numerically in Table 11c. Fig. 16 compares the ratcheting and shakedown regions with the equivalents for the original Bree loading [1,4], and also with the in-phase or anti-phase case, [5–7].

7. Ratchet strains

The ratchet strain (normalised by the yield strain $\varepsilon_y = \sigma_y/E$) is given by $\varepsilon_{ratchet} = 2Y(a-f)$ for all ratchet regions except R1f for which the ratchet strain is $\varepsilon_{ratchet} = 4aY$. The parameters a and f may be found using Tables 1–8 for each of the eight ratchet regions separately. The results are plotted against Y for a range of X values in Fig. 17.

8. Cyclic plastic strains

In the cyclic plasticity regions, P1 and P2, the cyclic plastic strain ranges are given by $\varepsilon_L = 2Y(0.5+g)$ on the left-hand surface, and $\varepsilon_R = 2Y(0.5-c)$ on the right-hand surface. Note that these are ranges, not amplitudes. (Recall that the two surfaces differ because the right-hand side is the positive side of the secondary bending load). The parameters c and g have been found numerically by applying the rules of §3.

It may sometimes be overlooked that a structure undergoing ratcheting will also be subject to cyclic plastic straining. This is significant because ratcheting *per se* is not a failure mechanism, but a deformation mechanism. Ratcheting effectively becomes a failure mechanism if the deformation exceeds some limit, such as a material ductility or a deformation limit based on functionality. However, fatigue damage will accumulate in the presence of cyclic plastic straining and this may be structurally limiting rather than the ratcheting.

Definition of the cyclic plastic strain range is ambiguous in the presence of ratcheting: it may be defined as the difference between the maximum plastic strain and either the immediately preceding minimum strain or the immediately following minimum strain. The two definitions differ by the ratchet strain. Here the convention has been adopted to use the second of these definitions, i.e., the smaller. Using this definition the cyclic plastic strain ranges in ratcheting regions are given by $\varepsilon_L = 2Y(0.5+d)$ on the left-hand surface, and $\varepsilon_R = 2Y(0.5-c)$ on the right-hand surface, except that the cyclic plastic strain ranges are zero on the right-hand surfaces in regions R1a and R1b, and on both surfaces in region R1f. The parameters d and c may be found using Tables 1–8 for each of the eight ratchet regions separately.

Fig. 18a plots the cyclic plastic strain range on the left-hand surface against Y for a range of X values, whilst Fig. 18b does the same for the right-hand surface. These plots cover both the P-regions and the R-regions. (All strains are normalised by the yield strain $\varepsilon_y = \sigma_y/E$).

The definition of cyclic plastic strain range used here in the ratcheting regions should be borne in mind if these results are used in a fatigue assessment. The correct strain range to use in a fatigue assessment is debateable. To guarantee conservatism, the ratchet strain should be added to the cyclic plastic strain range presented here before entering the fatigue endurance law.

Table 11c
Boundary curves on the ratchet diagram (Fig. 15): Numerical.

Curve GAB		Curve JAD	
X	Y	X	Y
0.01	196	0	2
0.05	36.4	0.1	2.0013
0.1	16.7	0.2	2.0059
0.15	10.3	0.3	2.0144
0.2	7.20	0.4	2.0282
0.25	5.40	0.45	2.0375
0.3	4.25	0.5127	2.0520
0.35	3.46	0.55	2.0595
0.4	2.88	0.6	2.0707
0.45	2.46	0.65	2.0834
0.5	2.12	0.7	2.0979
0.5127	2.052	0.75	2.1145
0.55	1.855	0.8	2.1339
0.6	1.617	0.8453	2.1547
0.65	1.401		
0.667	1.333		

9. Conclusion

The complete analytic solution for the positive out-of-phase cycling variant of the Bree problem has been derived. Fig. 15 shows the resulting ratcheting, shakedown and cyclic plasticity regions on the X, Y diagram. Fig. 16 compares the ratcheting and shakedown regions obtained for positive out-of-phase cycling with the original Bree loading [1,4], and also with the in-phase or anti-phase case, [5–7].

In the region where design code limits on primary loading are likely to be respected, i.e., $X<2/3$, the ratchet boundary for out-of-phase cycling lies between the original Bree and the in-phase cycling cases, being above the former and below the latter. However, in this region the shakedown boundary for out-of-phase cycling coincides with that for in-phase cycling, both lying below that for the original Bree case.

In the region where design codes limits on primary loading are likely to be violated, $X>2/3$, the out-of-phase results coincide with the original Bree solution: the shakedown region abuts a ratcheting region with no intervening plastic cycling region. This contrasts with the in-phase (or anti-phase) cycling case which is considerably more benign. However, the validity of this more benign solution depends upon an exact phase relation between the two loading cycles which is improbable in engineering practice. The out-of-phase case considered here is the more generic condition.

The present results reinforce the robustness of the original Bree solution as the basis of guidance in design codes of general applicability, i.e., the Bree ratchet boundary is conservative compared with the out-of-phase solution. The only minor exception to this is in regard to cyclic plasticity region P1 which lies below the Bree shakedown line. Thus, for X of, say, 0.5, cyclic plasticity may occur for out-of-phase loading at a secondary load amplitude which Bree would imply should shakedown to elastic cycling ($1.5<Y<2$). This may be significant as regards crack initiation through fatigue.

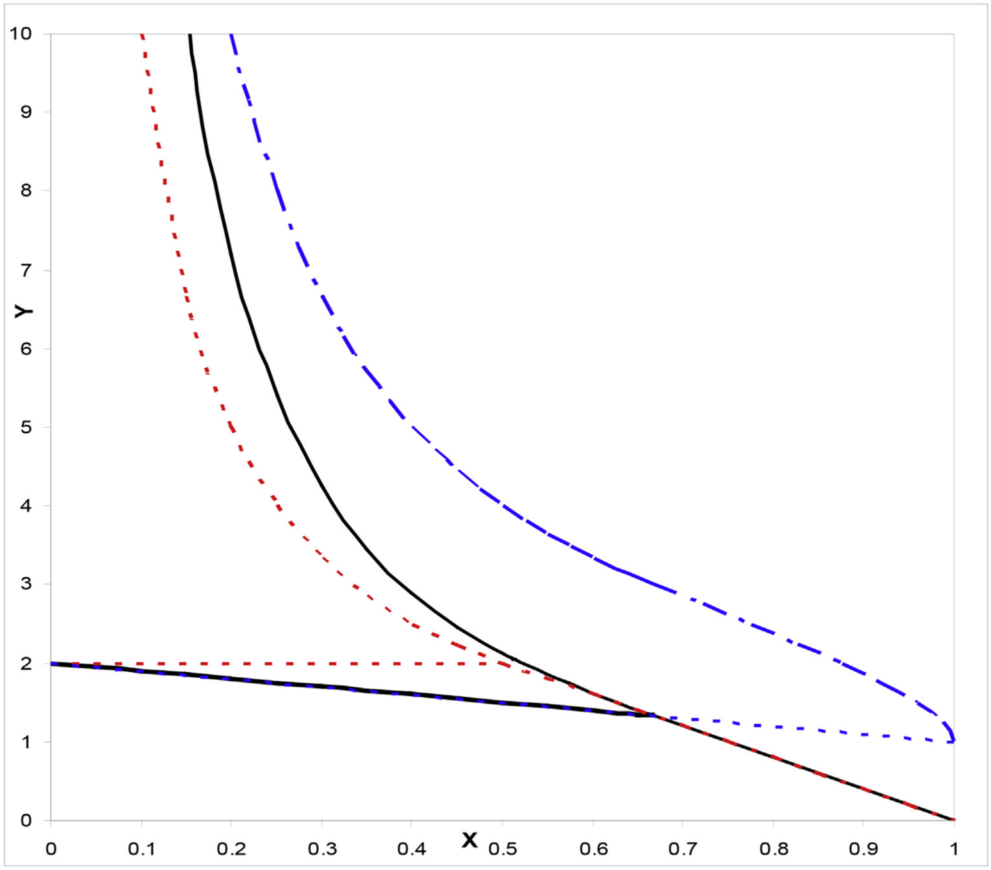

Fig. 16. Ratchet and Shakedown Regions for Out-of-Phase Cycling (black continuous) Compared With Original Bree Loading (red dashed) and In-Phase or Anti-Phase Cycling (blue dash-dotted).(For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

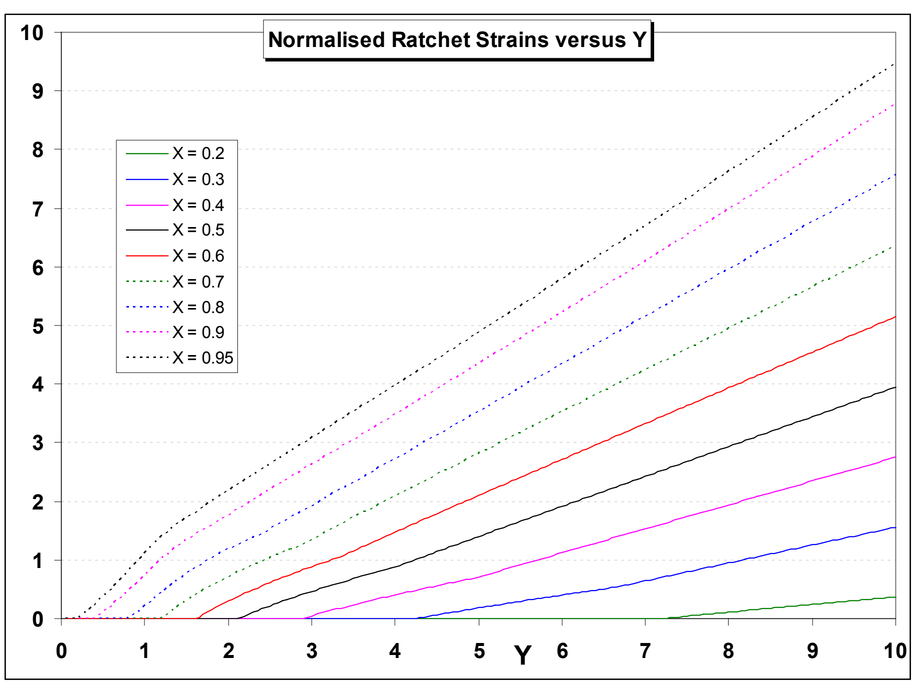

Fig. 17. Normalised ratchet strains versus Y for various X.

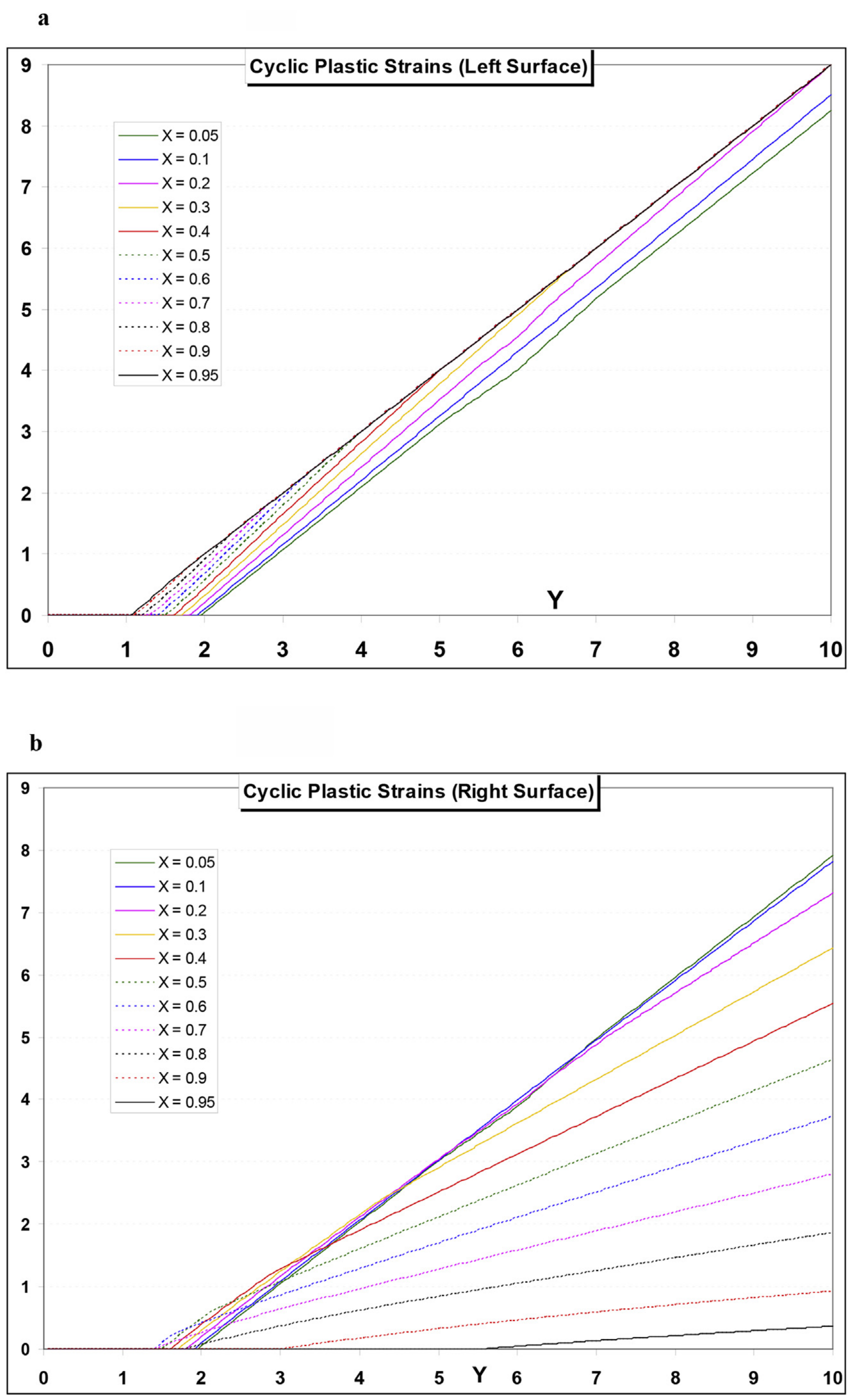

Fig. 18. (a) Cyclic plastic strain (left surface). (b) Cyclic plastic strain (right surface).

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