Integrating the Creep Relaxation Equation

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There is perennial confusion regarding the correct way to integrate the relaxation equation which this note aims to clarify.

R5 requires the use of strain hardening so the correct integration equation to use is,

$$\dot{\sigma} = -\frac{E}{Z}\left(\dot{\varepsilon}_c(\sigma, \varepsilon_c) - \dot{\varepsilon}_c(\sigma^P, \varepsilon_c)\right) \qquad (1)$$

where dots denote time derivatives. The stresses will generally be Mises quantities. The stress σ is the total (primary plus secondary) stress, whereas σ^P is the primary stress alone.

- In R5V2/3 applications, E in (1) would be replaced by the modified modulus, $\bar{E} = 3E/2(1+\nu)$, the stress σ would be the total stress at the assessment point whilst the primary stress σ^P would be interpreted as the rupture reference stress. (This interpretation ignores redistribution, consistent with the R5V2/3 approach of basing the assessment on the point stresses alone).
- In R5V4/5 applications, E in (1) would generally be the ordinary Young's modulus, though which modulus is used does not really matter so long as the elastic follow-up factor, *Z*, is calculated consistently. The stress σ would be interpreted as the total, or pseudo, reference stress, whilst σ^P would be interpreted as the primary reference stress (as opposed to the rupture reference stress).

The crucial feature of Eqn.(1) is that both terms on the RHS are evaluated at the same accumulated creep strain, ε_c. Formally this creep strain can be written,

$$\varepsilon_c = \int_0^t \dot{\varepsilon}_c(\sigma(t'), \varepsilon_c(t'))dt' \qquad (2)$$

which displays explicitly the dependence of both the stress and the strain upon time. However, Eqn.(2) cannot be integrated alone. Eqn.(1) must also be used to permit the creep strain, and the relaxed stress, to be found. Similarly the second term in (1) cannot be integrated alone because it does not depend only upon the primary stress but also upon the secondary stress via the strain ε_c. Consequently, whilst it may be tempting to write the integral of (1) as $\Delta\sigma = -E\left(\Delta\varepsilon_c - \Delta\varepsilon_c^P\right)/Z$, neither of the strain increments, $\Delta\varepsilon_c$ nor $\Delta\varepsilon_c^P$, can be evaluated in isolation. Both implicitly involve the whole relaxation curve.

The purpose of the second term on the RHS of (1) is to ensure that the stress becomes asymptotic to σ^P after a sufficiently long time. The stress cannot drop below σ^P. Note that it is essential that both the terms on the RHS of (1) are evaluated at the same creep strain, ε_c, in order that the two terms should cancel when the total stress approaches the primary stress, $\sigma \to \sigma^P$. If they were evaluated at different strains, the two terms would not cancel when $\sigma = \sigma^P$ and hence the stress asymptote would be incorrect.

If you are fortunate enough to have been given the creep deformation law in an explicitly strain dependent form, i.e., by specifying the instantaneous strain rate as an explicit function of stress and creep strain, $\dot{\varepsilon}_c = f(\sigma, \varepsilon_c)$, then it is straightforward to integrate numerically the combined equations (1) and (2). Looping over a sufficiently large number of sufficiently small time increments, Δt, the stress and strain are updated on each increment,

$$\sigma \to \sigma - \frac{E}{Z}\left(f(\sigma, \varepsilon_c) - f(\sigma^P, \varepsilon_c)\right)\Delta t \qquad (3a)$$

$$\varepsilon_c \to \varepsilon_c + f(\sigma, \varepsilon_c)\Delta t \qquad (3b)$$

More sophisticated numerical integration schemes are possible, but all should converge to the same results given sufficiently small time steps. We will not address these purely numerical issues here, but note,

- Using equal time steps will generally be very poor, especially if relaxation starts from the virgin state (zero creep strain). Time steps often need to be several orders of magnitude smaller near $\varepsilon_c = 0$ than later on. Simply increasing the time step size by some constant factor on every increment, $\Delta t_i = \lambda \Delta t_{i-1}$, can be effective.
- Strain rates may actually be divergent at $\varepsilon_c = 0$ requiring coding which avoids the singularity. Often it is possible to derive an algebraic expression for the creep strain at very short times, perhaps of the form $\varepsilon_c = A t^x \sigma^y$ where $x < 1$, hence avoiding the use of (3a,b) on the first increment.

However, the more tricky case is when creep strains have been supplied as a function of stress and time, rather than stress and strain (probably from constant load tests). Hence the input creep data is in the form of fits to the creep curves, Figure 1, i.e.,

$$\varepsilon_c = G(\sigma, t) \tag{4}$$

It is crucial to realise that Eqn.(4) only gives the correct creep strain if the stress is constant. So how can it be applied to calculate stress relaxation? This requires an additional assumption, specifically a hardening law – namely strain hardening in R5. So the extra hypothesis we need is…

Strain Hardening: *Regardless of the stress-strain history, the instantaneous creep strain **rate** at a point equals that of a monotonic constant load test at the same stress **and the same creep strain**.*

In principle, the way in which the strain hardening law can be used together with the time-dependent formulation of the test data, (4), is to invert (4) to give time in terms of creep strain,

$$t = h(\sigma, \varepsilon_c) \tag{5}$$

Referring to Figure 1, Eqn.(5) just consists of reading off the time (x-axis) corresponding to a given strain (y-axis) and a given stress (which specifies which curve). Even if a closed form expression for the inverse function is difficult or impossible to derive, it is simple in principle to do the inversion numerically.

The strain rate is given by,

$$\dot{\varepsilon}_c = \frac{\partial G}{\partial t} = g(\sigma, t) \tag{6}$$

which is also simple to derive either algebraically or numerically given G. Substituting (5) into (6) gives the strain rate in explicitly strain hardening form,

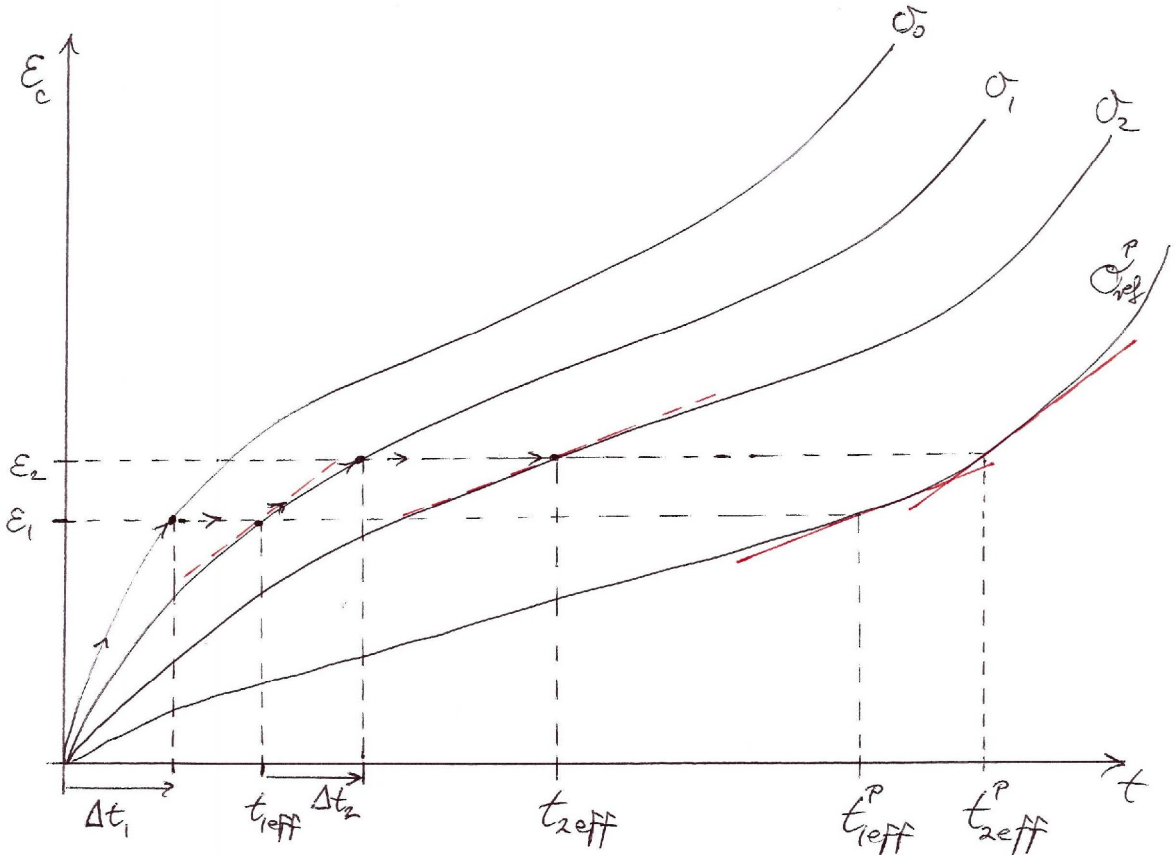
$$\dot{\varepsilon}_c = g(\sigma, h(\sigma, \varepsilon_c)) \tag{7}$$

The integration algorithm is thus just (3a,b), as before, but with the strain hardening function now given by $f(\sigma, \varepsilon_c) = g(\sigma, h(\sigma, \varepsilon_c))$, hence,

$$\sigma \to \sigma - \frac{E}{Z}\left(g(\sigma, h(\sigma, \varepsilon_c)) - g\left(\sigma^P, h\left(\sigma^P, \varepsilon_c\right)\right)\right)\Delta t \tag{8a}$$

$$\varepsilon_c \to \varepsilon_c + g(\sigma, h(\sigma, \varepsilon_c))\Delta t \tag{8b}$$

For people who prefer a graphical interpretation to the algebraic interpretation, (8a,b), consider Figure 1…



- After the first time increment, Δt_1, the stress has dropped from the initial σ_0 to σ_1, and the strain rate is given by the slope of the σ_1 curve evaluated at the strain, ε_1. This strain rate, $\dot{\varepsilon}_1$, can be obtained as the slope of the constant stress curve $\varepsilon_c = G(\sigma_1, t_{1eff})$ at the effective time t_{1eff}, defined by the requirement $\varepsilon_1 = G(\sigma_1, t_{1eff}) = G(\sigma_0, \Delta t_1)$.
- The next creep strain is estimated as $\varepsilon_2 = \varepsilon_1 + \dot{\varepsilon}_1 \Delta t_2$.
- The primary strain rate, $\dot{\varepsilon}_1^P$, which defines the second term on the RHS of Eqn.(1), or Eqn.(8a), is obtained as the slope of the σ^P constant stress curve at the effective time t_{1eff}^P, defined by the requirement $\varepsilon_1 = G(\sigma^P, t_{1eff}^P) = G(\sigma_1, t_{1eff})$.
- The relaxed stress is estimated as $\sigma_2 = \sigma_1 - \frac{E}{Z}(\dot{\varepsilon}_1 - \dot{\varepsilon}_1^P)\Delta t_2$.
- After the second time increment, Δt_2, the strain rate is given by the slope of the σ_2 curve evaluated at the strain, ε_2. This strain rate, $\dot{\varepsilon}_2$, can be obtained as the slope of the constant σ_2 stress curve at the effective time t_{2eff}, defined by the requirement $\varepsilon_2 = G(\sigma_2, t_{2eff}) = G(\sigma_1, t_{1eff} + \Delta t_2)$.
- The primary strain rate, $\dot{\varepsilon}_2^P$, which defines the second term on the RHS of Eqn.(8a) for the next increment, is obtained as the slope of the σ^P constant stress curve at the effective time t_{2eff}^P, defined by the requirement $\varepsilon_2 = G(\sigma^P, t_{2eff}^P) = G(\sigma_2, t_{2eff})$.

…and so on. Note that this graphical prescription becomes equivalent to Eqns.(8a,b) only in the limit of infinitesimal time steps. The link between the two descriptions is simply that the inverse function h, evaluated at the instantaneous stress and strain, equals the effective time,

$$t_{eff} \equiv h(\sigma, \varepsilon_c) \qquad (9)$$