Formulation of the Continuum Linear Elasticity Problem in 2D and the General Solution of the Axisymmetric Problem (Small Strain / Small Displacement Theory)

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The 3D formulation is discussed in another Note. In 2D plane stress the equations given there reduce to,

(A)Equilibrium: $\sigma_{x,x} + \tau_{,y} = 0$, $\sigma_{y,y} + \tau_{,x} = 0$ (assuming no body forces) (1)

(B)Hooke's Law: $E\varepsilon_x = \sigma_x - \nu\sigma_y$, $E\varepsilon_y = \sigma_y - \nu\sigma_x$, $G\gamma = \tau$ (2)

(C)Definition of Strains: $\varepsilon_x = u_{x,x}$, $\varepsilon_y = u_{y,y}$ $\gamma = 2\varepsilon_{xy} = u_{x,y} + u_{y,x}$ (3)

The plane strain problem requires no extra effort. In plane strain there is no change to the equations (1) and (3), but the stress-strain relations become,

$$E\varepsilon_x = \sigma_x - \nu\sigma_y - \nu\sigma_z, \qquad E\varepsilon_y = \sigma_y - \nu\sigma_x - \nu\sigma_z, \qquad E\varepsilon_z = \sigma_z - \nu\sigma_x - \nu\sigma_y = 0 \tag{2b}$$

The latter gives $\sigma_z = \nu(\sigma_x + \sigma_y)$, so that the first two become,

$$E\varepsilon_x = (1-\nu^2)\sigma_x - \nu(1+\nu)\sigma_y \quad \text{and} \quad E\varepsilon_y = (1-\nu^2)\sigma_y - \nu(1+\nu)\sigma_x \tag{2c}$$

But defining $E' = \frac{E}{1-\nu^2}$ and $\nu' = \frac{\nu}{1-\nu}$ these become, (2d)

$$E'\varepsilon_x = \sigma_x - \nu'\sigma_y \quad \text{and} \quad E'\varepsilon_y = \sigma_y - \nu'\sigma_x \tag{2e}$$

These are formally identical to the plane stress equations, (2). Hence, the solution to any 2D plane stress problem also provides the solution to the corresponding 2D plane strain problem simply by making the replacements $E \to E'$ and $\nu \to \nu'$.

In the Note on the 3D problem, it was found that the problem could be formulated either in terms of displacements or in terms of the stresses and strains without explicit use of the displacements. However, the later formulation was complicated because it involved six second order differential equations in the strains, the compatibility equations. In 2D, unlike 3D, formulation directly in terms of stresses and strains is simple because there is only one compatibility equation. It is,

$$\varepsilon_{x,yy} + \varepsilon_{y,xx} = 2\varepsilon_{xy,xy} = \gamma_{,xy} \tag{4}$$

Hence, the system of equations to be solved comprises Equs.(1), (2) and (4), a total of six equations in the six unknown quantities: $\sigma_x, \sigma_y, \tau, \varepsilon_x, \varepsilon_y, \gamma$.

A further simplification is possible by introducing a Airy function, φ. This acts like a 'potential' function for the stress field. The stresses are given in terms of the Airy function by,

(Cartesian coordinates): $\sigma_x = \varphi_{,yy} \qquad \sigma_y = \varphi_{,xx} \qquad \sigma_{xy} = \tau = -\varphi_{,xy}$ (5)

In terms of the Airy function, the equations of equilibrium, Equs.(1), become identities and we need not worry about them any more. [In fact, the existence of an Airy function obeying (5) is ensured by the equilibrium equations. The existence of a function φ satisfying Equs.(5) is the integrability condition for Equs.(1)].

The system of equations to be solved now comprises Equs.(2), (4) and (5). Substituting (5) into (2), and substituting the resulting expressions for the strains into (4) gives,

$$\nabla^4 \varphi = 0 \tag{6}$$

where $\nabla^4 = \left(\nabla^2\right)^2 = \left(\partial_x^2 + \partial_y^2\right)^2$. Equ.(6) is called the biharmonic equation. This shows that the 2D problem reduces to just one equation in just one unknown function, the Airy function, φ. However, it is a differential equation of fourth order. If we can solve for the Airy function, the stresses will be given by Equs.(5), and the strains by Equs.(2).

The General 2D Axisymmetric Problem

Axisymmetry naturally calls for the equations to be expressed in cylindrical (2D) polars. The Laplacian operator in polars is,

$$\nabla^2 = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 \tag{7}$$

For an axisymmetric problem, the biharmonic operator is thus,

$$\nabla^4 = \left(\partial_r^2 + \frac{1}{r}\partial_r\right)^2 \tag{8}$$

Expanding (8), the equation obeyed by an axisymmetric Airy function is thus,

$$\left[\partial_r^4 + \frac{2}{r}\partial_r^3 - \frac{1}{r^2}\partial_r^2 + \frac{1}{r^3}\partial_r\right]\varphi = 0 \tag{9}$$

It is straightforward to show by substitution that the general solution to (9) is,

$$\varphi = A + Br^2 + C\log r + Dr^2 \log r \tag{10}$$

[NB: Because Equ.(9) is an ordinary DE of fourth order, the GS will contain only four arbitrary constants, i.e. four linearly independent solutions. Hence Equ.(10) must be the most general solution].

In the general 2D case (i.e. not axisymmetric) the stresses are given in terms of the Airy function in polar coordinates by,

$$\sigma_r = \frac{1}{r^2}\varphi_{,\theta\theta} + \frac{1}{r}\varphi_{,r} \quad \text{and} \quad \sigma_\theta = \varphi_{,rr} \quad \text{and} \quad \tau_{r\theta} = \frac{1}{r^2}\varphi_{,\theta} - \frac{1}{r}\varphi_{,r\theta} \qquad (11)$$

In the axisymmetric case, substitution of the solution (10) in to (11) gives the stresses,

$$\sigma_r = 2B + \frac{C}{r^2} + D[1 + 2\log r] \quad \text{and} \quad \sigma_\theta = 2B - \frac{C}{r^2} + D[3 + 2\log r] \qquad (12)$$

The shear stress is zero, of course. Note that the constant A does not appear in Equs.(12). The Airy function, being a type of ‘potential’ has an arbitrary datum zero.

Example: Thick Pressurised Pipe Under Internal and External Pressure

It suffices to consider the solution with D = 0. The boundary conditions are that the radial stress equals (minus) the pressure at the inner and outer surfaces. This allows the constants B and C to be found from,

$$2B + \frac{C}{R_i^2} = -P_i \quad \text{and} \quad 2B + \frac{C}{R_0^2} = -P_0$$

Hence,

$$2B = \frac{P_i R_i^2 - P_o R_o^2}{R_o^2 - R_i^2} \quad \text{and} \quad C = -\frac{\Delta P R_i^2 R_o^2}{R_o^2 - R_i^2}$$

where $\Delta P = P_i - P_o$. Hence, the hoop and radial stresses at any radius r are,

$$\sigma_\theta = \frac{P_i R_i^2 - P_o R_o^2}{R_o^2 - R_i^2} + \frac{\Delta P R_i^2 R_o^2}{(R_o^2 - R_i^2) r^2} \quad \text{and} \quad \sigma_r = \frac{P_i R_i^2 - P_o R_o^2}{R_o^2 - R_i^2} - \frac{\Delta P R_i^2 R_o^2}{(R_o^2 - R_i^2) r^2}$$

Hence, for example, the hoop stress with no external pressure becomes,

($P_o = 0$):

$$\sigma_\theta = P_i \left(\frac{R_i^2 R_o^2}{R_o^2 - R_i^2}\right)\left(\frac{1}{R_o^2} + \frac{1}{r^2}\right)$$

For external pressure alone the hoop stress is,

($P_i = 0$):

$$\sigma_\theta = -P_o \left(\frac{R_i^2 R_o^2}{R_o^2 - R_i^2}\right)\left(\frac{1}{R_i^2} + \frac{1}{r^2}\right)$$

In the latter case, if we let $R_o \to \infty$ the problem becomes that of an infinite plate containing a circular hole with a remote equibiaxial applied stress of $\sigma_{r\infty} = -P_0$. Giving, (Hole in plate: equibiaxial tension): $\sigma_\theta = \sigma_{r\infty}\left(1 + \frac{R_i^2}{r^2}\right)$

So, the SCF at the surface of the hole where r = R_i, is determined to be 2.

A further exercise is to drop the requirement for axisymmetry, starting again from Equ.(7), and to derive the solution for a hole in a plate subject to remote uniaxial tension. This exercise is carried out in a separate Note on this site, where it is determined that the SCF at the surface of the hole is then 3.