Chapter 5C – Cosmic Geometry: Part 2
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1. Introduction
We continue here our outline of the basic geometrical properties of cosmological spacetimes which we began in Chapter 5B. A joint Index to both Chapters is as follows:-

- Newtonian Cosmologies (Ch.5B §2);
- The formulation of general relativity (Ch.5B §3);
- Cosmological models (Friedman-Robertson-Walker space-times) (Ch.5B §4);
- The surprising observation that Newtonian cosmologies encompass the general relativistic cosmologies provided that pressure and the cosmological constant are zero (Ch.5B §3 & §5);
- Precisely what is meant by an "expanding universe" (Ch.5B §4 + Ch.5C);
- Precisely what is meant by an infinite universe, and why an infinite universe would always have been infinite, even at the Big Bang (Ch.5B §4.3);
- What is meant by the curvature of space and how it is related to whether the universe is finite or infinite (Ch.5B §4);
- Explanation of why I keep referring to the "size scale" of the universe, rather than simply "the size of the universe" (Ch.5B §4);
- What is the "Cosmological Constant" and how does it affect the possible cosmological models (Ch.5B §3 & §5);
- How the relationship between the age of the universe and the Hubble time depends upon the cosmological model (Ch.5B §6);
- The current consensus model (based on observational data) (Ch.5B §7);
- The cosmology of a universe filled with a pressurised fluid – how pressure causes gravity (Ch.5C);
- Demonstration that an expanding universe leads to a redshift, and that this redshift obeys Hubble's Law, at least to first order? or is it exact? (Ch.5C);
- The definitions of several different length measures and derivation of their relation to the redshift for different cosmological models (Ch.5C);
- In particular, the definition of the length scale in Hubble's Law v = HR which permits the "velocity" v = dR/dt to exceed c (Ch.5C);
- The size of the observable universe – and why this is greater than ct, but depends upon the cosmological model (Ch.5C);
- Why the horizon encompasses a decreasing amount of matter as we look further back in time – the Horizon Problem (Ch.5C);
- The topology of the universe – how a non-trivial topology could solve the "finite but unbounded" problem but with zero curvature (Ch.5C).

2. Redshifts and Distances
To recap, the most general metric satisfying the assumptions stated in Section 2 has the form,

$$ds^2 = c^2 dt^2 - R^2(t)dL_A^2 \tag{2.1}$$

where dL_A is a length element in a spatial sub-space. Since the factor of R(t) carries the units of length, dL_A is a sort of angular interval, hence the subscript. (For k > 0 it

is almost literally an angular interval, in the hypothetical 4D Euclidean embedding space). It can be written (amongst other ways) as,

$$dL_A^2 = d\psi^2 + S(\psi)^2\left(d\theta^2 + \sin^2\theta \cdot d\phi^2\right) \tag{2.2}$$

where, $S(\psi)$ is $\psi, \sin\psi$ or $\sinh\psi$ according to whether space is flat, positively curved or negatively curved (i.e., $k = 0, +1$ or -1).

2.1 The Redshift

We now show that radiation emitted and received by co-moving observers in a Robertson-Walker space-time is subject to a redshift (or a blue shift if the universe is contracting). Consider two co-moving observers, 1 and 2. By definition these observers are at rest in (ψ, θ, ϕ) co-ordinates, i.e. observer 1 is at $(\psi_1, \theta_1, \phi_1)$ at all times and observer 2 is at $(\psi_2, \theta_2, \phi_2)$ at all times. Thus, the "angular" separation between them, ΔL_A, is constant.

Suppose observer 1 emits a pulse of light towards 2 at time t_1. Thanks to the global time co-ordinate, t, we do not have to be concerned about the relativity of time between the two observers. The assumption of a global time co-ordinate makes cosmological kinematics easier than that of special relativity, surprisingly. Suppose observer 2 receives the pulse of light at time t_2. Imagine the path of the light beam being broken into a sequence of equal time periods dt. The light traverses an angular interval dL_A in each period dt, but note that the size of these dL_A will vary with time. Because light travels on the null cone, i.e. ds = 0, from (2.1) we have,

$$R(t)dL_A = cdt \tag{2.3}$$

Thus, if the universe is expanding monotonically over the period in question, the angular intervals dL_A corresponding to fixed time increments, dt, become smaller and smaller at later times. This is illustrated by,
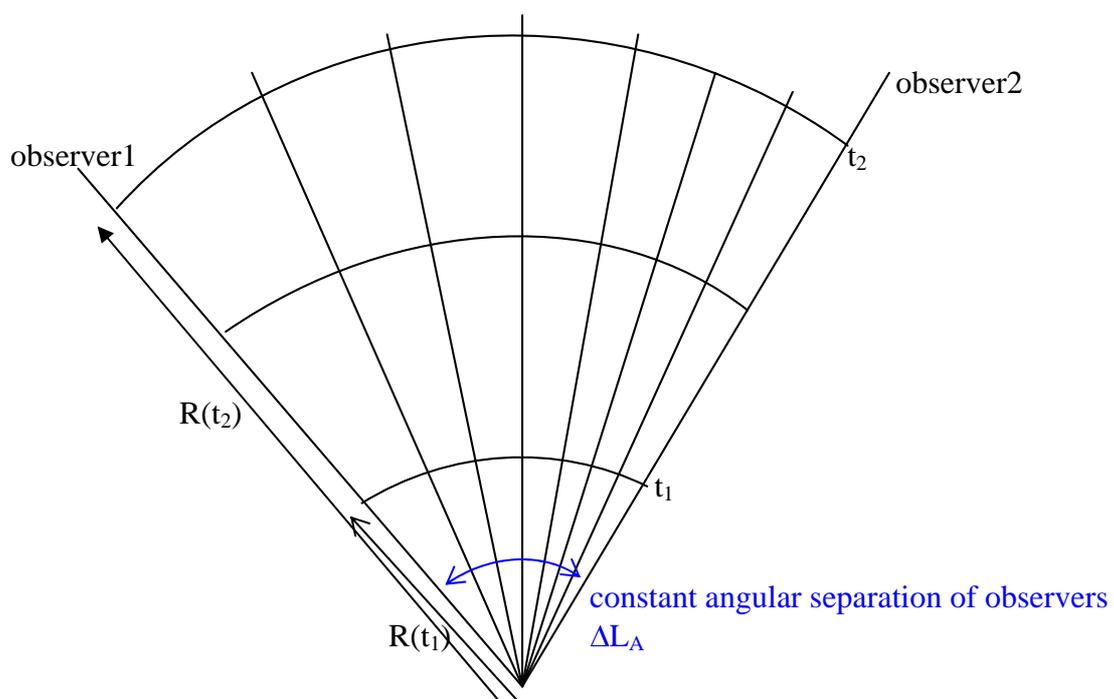
The total angular separation of the observers is obtained by summing over the unequal angular increments, i.e. from (6.3),

$$\Delta L_A = \int_{t_1}^{t_2} \frac{cdt}{R(t)} \tag{2.4}$$

If we consider another light pulse emitted by observer 1 at time $t_1 + \Delta t_1$, and received by observer 2 at time $t_2 + \Delta t_2$, then clearly,

$$\Delta L_A = \int_{t_1 + \Delta t_1}^{t_2 + \Delta t_2} \frac{cdt}{R(t)} \tag{2.5}$$

Note that because the angular separation of the two observers is constant, the LHS of (6.4) and (6.5) are the same.

Now we may imagine the two consecutive pulses to be the peaks in a sinusoidal wave train, i.e. the interval Δt_1 is the periodic time seen by observer 1, whilst the interval Δt_2 is the periodic time seen by observer 2. Subtracting (6.4) and (6.5) gives,

$$\int_{t2}^{t_2 + \Delta t_2} \frac{cdt}{R(t)} - \int_{t_1}^{t_1 + \Delta t_1} \frac{cdt}{R(t)} = 0 \tag{2.6}$$

But for time intervals small compared with the expansion rate of the universe, this gives simply,

$$\frac{\Delta t_2}{R(t_2)} = \frac{\Delta t_1}{R(t_1)} \Rightarrow \frac{\Delta t_2}{\Delta t_1} = \frac{f_1}{f_2} = \frac{\lambda_2}{\lambda_1} = \frac{R(t_2)}{R(t_1)} \tag{2.7}$$

where f_1 is the frequency seen by observer 1 and f_2 is the frequency seen by observer 2, and similarly λ is the wavelength of the radiation. Thus, in an expanding universe, the ratio in (2.7) is greater than unity and the radiation is red-shifted. We see that knowledge of the universe's size scale, R(t), at the two times suffices to determine the red-shift.

Astronomers defined the red-shift as,

$$z = \frac{\Delta \lambda}{\lambda} = \frac{\lambda_2 - \lambda_1}{\lambda_1} = \frac{\lambda_2}{\lambda_1} - 1 = \frac{R(t_2)}{R(t_1)} - 1 \tag{2.8}$$

In practical applications, we are "observer 2", detecting the radiation, and t_2 is the present time (t_{now}). If a cosmological model is assumed, giving the function R(t), an observation of the red-shift allows the time of emission to be found from (2.8), that is,

$$z_{observed} = \frac{R(t_{now})}{R(t_{emission})} - 1 \tag{2.9}$$

It follows that an observed red-shift can be translated into a time of emission by inversion of the function R(t), i.e.,

$$t_{emission} = R^{-1}\left(\frac{R_{now}}{1+z}\right)$$
(2.9b)

Explicit examples of (2.9b) will be discussed later for specific cosmologies.

2.2 Distance Measures D_{now} and D_{ltt}

In Section 2.1 we have talked about the (constant) angular separation of two co-moving observers, ΔL_A. But what is their actual separation? At this point we come hard up against the fact that there is no unique definition of the distance between two co-moving observers.

One obvious measure, since the speed of light is supposed to be constant, is the sum of the light path increments cdt that are required to send a signal from 1 to 2. Thus, the "light transit time" distance, D_{ltt}, is defined as,

$$D_{ltt} = c\Delta t$$
(2.10)

This is a very convenient definition of a distance if we happen to know what the light transit time is. However, if our two co-moving points in space are defined by their (constant) ψ, θ, ϕ co-ordinates (i.e. we are given their "angular" separation, ΔL_A), then it is not immediately obvious what the light transit time is. Since the distance apart of the two points at time t_1 is $R(t_1)\,\Delta L_A$ and this increases to $R(t_2)\,\Delta L_A$ at time t_2, we have,

$$R(t_1)\Delta L_A < c\Delta t < R(t_2)\Delta L_A$$
(2.11)

but what is $c\Delta t$ exactly for a given ΔL_A?

Before answering this question, we turn to a different definition of distance. We have already appealed to it in the above discussion. At any time t' the instantaneous distance between our two points at a given "angular" spacing, ΔL_A, is simply $R(t')\Delta L_A$. Thus if we choose t' to be the time now, we have the definition of "distance now", i.e.,

$$D_{now} = R(t_{now})\Delta L_A$$
(2.12)

D_{now} is a very convenient measure of distance if we happen to know the "angular" distance between the two points in question (and we also know the current size scale, $R(t_{now})$, of the universe – presumably through some cosmological model). However, if the two points have been defined by specifying the light transit time connecting them, how do we find D_{now} in that case? We see that this is essentially the problem we had with D_{ltt}, but in reverse.

In both cases the answer is provided by (2.4). This allows us to find the "angular" interval ΔL_A between points connected by a null geodesic (light) over the time interval [t1, t2]. Hence the distance D_{now} can be found for a pair of points connected by light over a specified time interval. By inverting (6.4) we can also find the light transit time interval corresponding to a given "angular" separation, ΔL_A. This is best illustrated by examples, below.

2.3 D_{now} and D_{ltt} for Flat Space (k = 0) and Λ = 0 and Matter Dominance
As discussed in Chapter 5B Section 5, this cosmological model has the solution,

$$R(t) = \tilde{A}t^{\frac{2}{3}} \Rightarrow R(t) = R_0\left(t/t_0\right)^{\frac{2}{3}} \tag{2.13}$$

where $\tilde{A}$ is given by Chapter 5B Equ.(2.9), $\tilde{A} = \left(6\pi G\rho_0 r_0^3\right)^{\frac{1}{3}}$. However, for sufficiently small times, R(t) behaves like (2.13) for all cosmological models for which R = 0 at a finite time, assuming matter dominance. Hence (2.13) is a generic behaviour for early times assuming matter dominance[1]. In (2.13), t_0 is some arbitrary time when the size scale is R_0. We can now carry out the integral in (2.4) explicitly, to give,

$$\Delta L_A = \frac{3c}{R_0}t_o^{\frac{2}{3}}\left(t_2^{\frac{1}{3}} - t_1^{\frac{1}{3}}\right) \tag{2.14}$$

We are usually interested in the case when the later time is now ($t_2 = t_{now}$). Choosing the arbitrary time t_0 to be t_{now} also, we get,

$$\Delta L_A = \frac{3c}{R_{now}}t_{now}\left(1-\left(\frac{t_1}{t_{now}}\right)^{\frac{1}{3}}\right) \tag{2.15}$$

Thus, (2.15) gives the angular interval ΔL_A between two points linked by a light ray between times t_1 and t_{now}. Hence, the distance now is, from (2.12),

$$D_{now} = 3ct_{now}\left(1-\left(\frac{t_1}{t_{now}}\right)^{\frac{1}{3}}\right) \tag{2.15b}$$

A special case of (2.15) is the maximum "angular" separation between two points which are causally linked (now). This is the separation corresponding to light travelling since the Big Bang, obtained by putting $t_1 = 0$, hence,

Observable Universe: $\Delta L_A = \dfrac{3c}{R_{now}}t_{now} \Rightarrow D_{now} = \Delta L_A R_{now} = 3ct_{now}$ (2.16)

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[1] This is rather a contradiction, since at sufficiently early times the universe will be radiation dominated.

Thus, we have the result that, for these cosmological models, the size of the observable universe (in terms of the distance measure D_{now}) is just three times ct_{now}, where t_{now} is the current age of the universe. Note that the factor of three originates from the dependence of R(t) on a power of time in Equ.(2.13).

One may reasonably ask how the boundary of the observable universe can be at a distance of $3ct_{now}$ given that light travels at speed c and has been doing so for the time t_{now}. The answer is that it depends upon what distance measure you choose to employ. It we choose the light transit time based distance measure, D_{ltt}, then by definition the observable universe is of size $D_{ltt} = ct_{now}$. However, it is often more useful to know the size of the observable universe now, that is the size measure D_{now}. The reason the latter is larger is that, in addition to the distance covered by the light, space has also expanded during the time that the light was in transit. As a check we can derive the key equation, (2.4), in another way.

Let the light start at time t_1 and be observed at time t_{now}, as above. Consider some intermediate time t'. In the interval dt' the light travels cdt'. However, between time t' and when the light is observed the universe will expand from R(t') to R(t_{now}). Hence, the distance cdt' travelled by the light at around t' will be stretched to a distance [R(t_{now})/R(t')]cdt' by the time of the observation. To get the total "distance now" we must add all these stretched contributions, i.e.,

$$D_{now} = \int_{t_1}^{t_{now}} cdt' \cdot \frac{R(t_{now})}{R(t')} \qquad (2.17)$$

But this is just the same as (2.4), by virtue of the definition of D_{now}, see (2.12).

The Varying Causal Horizon

(2.16) shows that the distance D_{now} to the boundary of the observable universe increases linearly with time. But, of course, the universe itself is expanding non-linearly with time, as given by (2.13). Consequently, the observable universe will encompass a changing amount of material as the expansion proceeds. The best way of seeing this is from (2.16) which shows that the "angular" distance to the boundary of the observable universe at any time t is,

Causally Connected Region: $\qquad \Delta L_A = \dfrac{3ct}{R(t)} = \dfrac{3ct}{\tilde{A}t^{2/3}} = \dfrac{3c}{\tilde{A}} t^{1/3} \qquad (2.18)$

If the universe had been expanding linearly with time, then this "angular" distance to the observable boundary would have been independent of time. In this case, a given "angular" interval would always have contained exactly the same matter. However, for any cosmological model exhibiting a Big Bang the actual behaviour is as given by (2.18) for sufficiently early times (and for all times in a flat universe with $\Lambda = 0$, assuming matter dominance). Thus, the observable universe is described by an angular interval which increases with time, and hence encompasses an increasing amount of material.

Of course, the "observable universe" is just another way of talking about the maximum extent of the region which is causally connected to its central point (us) at a given time. Since the "angular" interval (2.18) shrinks to zero at the Big Bang it follows that each particle of matter is causally connected to less and less surrounding matter as we approach the Big Bang (going backwards in time).

Is this another way of looking at reaction freeze-out by cosmic expansion rate? Check.

Finally, we have still not found explicitly the D_{ltt} distance for a given "angular" interval, ΔL_A. From (2.15) and (2.13) with $t_{now} = t_1 + \Delta t$ we find,

$$\Delta t = \left[\frac{\tilde{A}}{3c} \Delta L_A + t_1^{1/3} \right]^3 - t_1 \tag{2.19}$$

Hence, the distance D_{ltt} will in general depend upon the absolute time as well as upon the "angular" interval, ΔL_A, traversed. If $t_1 = 0$ then we get,

$$D_{ltt} = c\Delta t = \left[\frac{\tilde{A}}{3c} \Delta L_A \right]^3 \tag{2.20}$$

which is just the inverse of (2.18). It is rather surprising that the light transit time distance between two points depends upon the cube of their angular separation.

The Relationship of Dnow and Dltt to Redshift z
It is convenient to use our scaling factor $a(t) = R(t)/R_{now}$ in this section. The red-shift can thus be written, using (2.9),

$$z = \frac{R(t_{now})}{R(t)} - 1 = \frac{1}{a} - 1 \Rightarrow a = \frac{1}{1+z} \tag{2.21}$$

And from (2.13) we have, $a(t) = \left(\frac{t}{t_{now}} \right)^{2/3}$ and hence $1 + z = \left(\frac{t_{now}}{t} \right)^{2/3}$ (2.22)

which also yields the useful time-red-shift relation: $\quad t = \dfrac{t_{now}}{(1+z)^{3/2}}$ (2.22b)

Note that (2.22) and (2.22b) apply only for matter dominance and flat space, $k = 0$, with $\Lambda = 0$, or for sufficiently early times if k or Λ and non-zero. Actually, (2.22) and (2.22b) are rather inaccurate. The reason is that the term t_{now} in the numerator cannot really be interpreted as the present age of the universe. This stems from (2.13). This power law expression does not generally hold up until the present time. A more accurate time-red-shift relation is derived as (3.11) below.

Note that (2.22b) shows that signals from the Big Bang ($t = 0$), i.e. points on the horizon, will have a divergent red-shift, $z \to \infty$.

We can derive D_{now} in terms of z directly from (2.15b), i.e.,

$$D_{now} = 3ct_{now}\left[1-\left(\frac{t}{t_{now}}\right)^{1/3}\right] = 3ct_{now}\left[1-\frac{1}{\sqrt{1+z}}\right] \tag{2.23}$$

Thus, for a flat cosmological model (or whenever $R(t) \propto t^{2/3}$ is believed to be a good approximation) Equ.(2.23) can be used by astronomers to find the distance of an object from its measured red-shift, z. For example, objects with a red-shift of 3 will be halfway to the horizon (for this simple cosmological model with $k = 0$ and $\Lambda = 0$).

A general means of finding such relations exists which does not require explicit integration for D_{now} analogous to (2.15). This is obtained by noting that,

$$D_{now} = \int_{t}^{t_{now}} \frac{cdt}{a(t)} = \int_{\frac{1}{1+z}}^{1} \frac{cda}{a\dot{a}} \tag{2.24}$$

where the limits of the 'a' integral have been obtained from (2.21). We do need to know $\dot{a}$ in terms of a. However this is provided directly by the Friedman equation, Chapter 5B Equ.(5.2), without the need for integration:-

$$\dot{a}^2 = \frac{8\pi}{3}G\rho a^2 - \frac{kc^2}{R_{now}^2} + \frac{\Lambda c^2}{3}a^2 \tag{2.24b}$$

In this case, $k = \Lambda = 0$, and with matter dominance (constant ρR^3) this becomes,

$$\dot{a} = \frac{2}{3t_{now}\sqrt{a}} \tag{2.25}$$

so that (2.24) readily reproduces (2.23). In like manner we have, in general,

$$D_{ltt} = \int_{t}^{t_{now}} cdt = \int_{\frac{1}{1+z}}^{1} \frac{cda}{\dot{a}} \tag{2.26}$$

which, in this case, using (2.25), gives,

$$D_{ltt} = ct_{now}\left[1-\frac{1}{(1+z)^{3/2}}\right] \tag{2.27}$$

consistent with (2.22b).

2.4 Other Distance Measures: Real-Angle Distance and Luminosity Distance

3. Numerical Solutions for Curved Space ($k \neq 0$), With and Without Λ, ($P = 0$)

In Chapter 5B we described the qualitative features of the solutions to the Friedman equation for a pressureless fluid. However we did not attempt to find closed form solutions. In this Section we shall find numerical solutions of the Friedmann equation for arbitrary k, arbitrary cosmological constant, and arbitrary mass-energy density. We shall continue to assume matter dominance in this Section[2]. These numerical solutions yield R, or the red-shift, z, as functions of time. Working within these numerical solutions it is little extra work to evaluate the integrals which yield the distance measures D_{ltt}, D_{now}, etc. It is convenient to work in the red-shift, so we first re-cast the Friedmann equation in terms of z.

Recall that the Friedmann equation is [Chapter 5B, equ.()]:-

$$\dot{R}^2 = \frac{8\pi}{3}G\rho R^2 - kc^2 + \frac{\Lambda c^2}{3}R^2 \qquad (3.1)$$

The density parameter is $\Omega = \frac{\rho}{\rho_{crit}} = \frac{8\pi G\rho}{3H^2}$. This parameter can vary with time, but we can write its value now as $\Omega_0 = \frac{\rho_0}{\rho_{crit,0}} = \frac{8\pi G\rho_0}{3H_0^2}$. Using $\rho R^3 = \rho_0 R_0^3$ the first term in (3.1) is thus,

$$\frac{8\pi G\rho R^2}{3} = \frac{8\pi G\rho_0 R_0^3}{3R} = \frac{\Omega_0 H_0^2 R_0^3}{R} \qquad (3.2)$$

The second term in (3.1) can be written, using the definition of the curvature parameter given in Chapter 5B,

$$-kc^2 = H^2 R^2 \Omega_R = H_0^2 R_0^2 \Omega_{R0} \qquad (3.3)$$

where the second from follows from the fact that k is a constant. The last term in (3.1) can be written in terms of the dimensionless dark-energy parameter as follows,

$$\frac{\Lambda c^2 R^2}{3} = \Omega_\Lambda H^2 R^2 = \Omega_{\Lambda 0} H_0^2 R^2 \qquad (3.4)$$

where the second form follows because Λ is a constant. Note that (3.3) involves the size scale now, R_0, whereas (3.4) involves the time dependent R, and (3.2) involves both. Adding (3.2), (3.3) and (3.4) and dividing by R^2 gives,

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[2] This means that we need only deal with the first order Friedmann equation, as given in Chapter 5B Equ.(5.2), since the second order equation, Chapter 5B Equ.(5.1), is implied by the 'fluid equation' which, for P = 0 reduces to the matter dominance condition that ρR^3 is constant.

$$\left(\frac{\dot{R}}{R}\right)^2 = H_0^2\left\{\Omega_0\left(\frac{R_0}{R}\right)^3 + \Omega_{R0}\left(\frac{R_0}{R}\right)^2 + \Omega_{\Lambda 0}\right\} \qquad (3.5)$$

Note that whilst Ω, Ω_R and Ω_Λ are all time dependent, the quantities occurring in Equ.(3.5) are their values now, and hence are constants. The same is true of H_0.

From the definition of the red-shift, (2.21), we can replace R_0/R in (3.5) by $1 + z$, giving,

$$\left(\frac{\dot{R}}{R}\right)^2 = H_0^2\left\{\Omega_0(1+z)^3 + \Omega_{R0}(1+z)^2 + \Omega_{\Lambda 0}\right\} \qquad (3.6)$$

Moreover, by differentiating (2.21) we find that,

$$\frac{\dot{R}}{R_0} = -\frac{\dot{z}}{(1+z)^2} \qquad \text{and} \qquad \frac{\dot{R}}{R} = -\frac{\dot{z}}{(1+z)} \qquad (3.7)$$

So that (3.6) becomes,

$$\left(\frac{\dot{z}}{1+z}\right)^2 = H_0^2\left\{\Omega_0(1+z)^3 + \Omega_{R0}(1+z)^2 + \Omega_{\Lambda 0}\right\} \qquad (3.8)$$

Note that, when the universe is expanding, we will need to take the negative square-root of (3.8) to find $\dot{z}$. Thus, we find a first order differential equation for the red-shift,

$$\frac{dz}{dt} = -H_0\left\{\Omega_0(1+z)^5 + \Omega_{R0}(1+z)^4 + \Omega_{\Lambda 0}(1+z)^2\right\}^{1/2} \qquad (3.9)$$

The three terms, arising respectively from mass-energy (Ω_0), curvature (Ω_{R0}), and the cosmological constant (Ω_Λ), are distinguished in (3.9) by the different orders of the corresponding z, or R, dependence.

Before looking at numerical solutions of (3.9) we derive a commonly used relation between time and red-shift. If we confine attention to sufficiently early times, i.e. to sufficiently large red-shifts[3], then the first term in (3.9) is dominant giving,

$$\frac{dz}{dt} \approx -H_0\Omega_0(1+z)^{5/2} \qquad (3.10)$$

We now integrate (3.10) from time 0 to t, when the red-shift drops from ∞ to z, which gives,

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[3] But not so early that the assumption of matter dominance (zero pressure) breaks down.

$$t = \frac{2}{3} \cdot \frac{1}{H_0 \Omega_0^{\frac{1}{2}} (1+z)^{\frac{3}{2}}} \qquad (3.11)$$

Note that an inaccuracy has crept into (3.11) since, at the earliest times the assumption of matter dominance (zero pressure) upon which (3.9) is based, will no longer be correct. Because we integrate from $t = 0$ to derive (3.11), this will affect the time-red-shift relationship. More importantly, (3.11) only applies for large red-shifts ($z \gg 1$), because we have neglected the second and third terms in (3.9). Consequently it is not valid to put $z = 0$ in (3..11) and falsely conclude that $t_{now} = \frac{2}{3} \cdot \frac{1}{H_0 \Omega_0^{\frac{1}{2}}}$. The correct
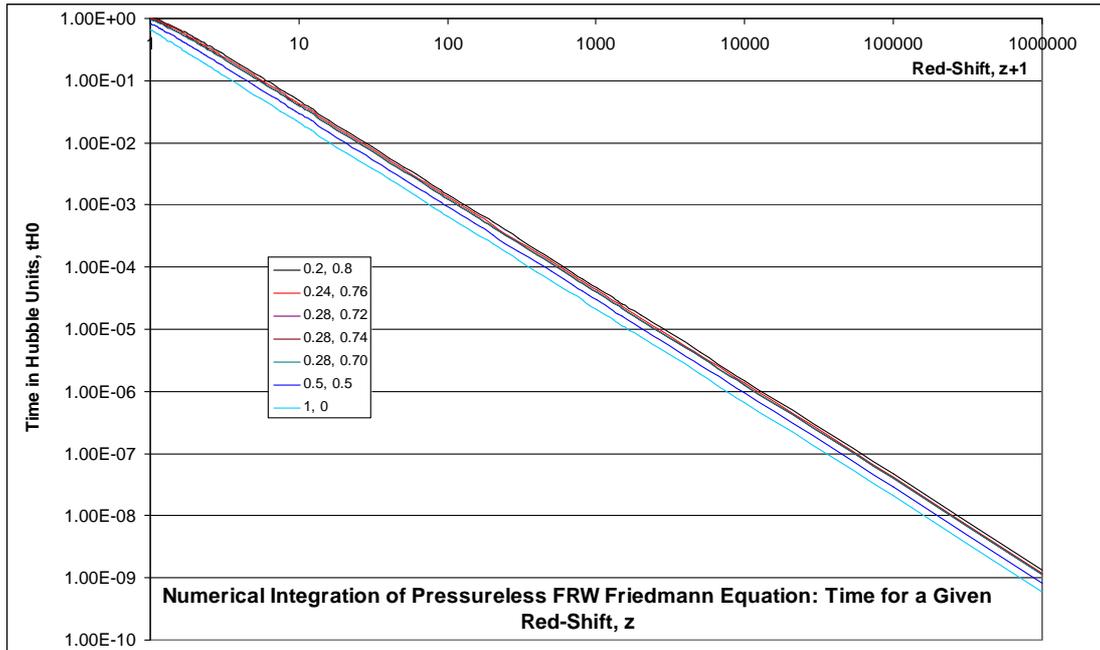
derivation of t_{now} can be found in Chapter 5B Section 6. It depends also upon $\Omega_{\Lambda 0}$, and possibly also upon Ω_{R0} if this is non-zero.

Numerical Solutions for z and R

| z	$\Omega_0 = 0.2$ $\Omega_{\Lambda 0} = 0.8$ $\Omega_{R0} = 0$	0.24 0.76 0	0.28 0.72 0	0.28 0.74 -0.02	0.28 0.7 0.02	0.5 0.5 0	1 0 0
1000000	1.32E-09	1.21E-09	1.12E-09	1.12E-09	1.12E-09	8.37E-10	5.91E-10
100000	4.69E-08	4.28E-08	3.96E-08	3.96E-08	3.96E-08	2.97E-08	2.10E-08
10000	1.49E-06	1.36E-06	1.26E-06	1.26E-06	1.26E-06	9.41E-07	6.66E-07
5000	4.21E-06	3.85E-06	3.56E-06	3.56E-06	3.56E-06	2.66E-06	1.88E-06
1000	4.71E-05	4.30E-05	3.98E-05	3.98E-05	3.98E-05	2.98E-05	2.10E-05
500	1.33E-04	1.21E-04	1.12E-04	1.12E-04	1.12E-04	8.41E-05	5.94E-05
100	1.47E-03	1.34E-03	1.24E-03	1.24E-03	1.24E-03	9.29E-04	6.57E-04
50	4.09E-03	3.74E-03	3.46E-03	3.46E-03	3.46E-03	2.59E-03	1.83E-03
20	1.55E-02	1.41E-02	1.31E-02	1.31E-02	1.31E-02	9.80E-03	6.93E-03
15	2.33E-02	2.13E-02	1.97E-02	1.97E-02	1.97E-02	1.47E-02	1.04E-02
10	4.08E-02	3.73E-02	3.45E-02	3.46E-02	3.45E-02	2.58E-02	1.83E-02
9	4.71E-02	4.30E-02	3.98E-02	3.99E-02	3.97E-02	2.98E-02	2.11E-02
8	5.52E-02	5.04E-02	4.66E-02	4.67E-02	4.65E-02	3.49E-02	2.47E-02
7	6.58E-02	6.01E-02	5.56E-02	5.58E-02	5.55E-02	4.17E-02	2.95E-02
6	8.03E-02	7.34E-02	6.79E-02	6.81E-02	6.77E-02	5.09E-02	3.60E-02
5	1.01E-01	9.24E-02	8.56E-02	8.59E-02	8.53E-02	6.41E-02	4.54E-02
4	1.33E-01	1.21E-01	1.12E-01	1.13E-01	1.12E-01	8.42E-02	5.96E-02
3	1.84E-01	1.69E-01	1.56E-01	1.57E-01	1.56E-01	1.18E-01	8.33E-02
2	2.80E-01	2.57E-01	2.39E-01	2.40E-01	2.37E-01	1.80E-01	1.28E-01
1.5	3.63E-01	3.34E-01	3.11E-01	3.13E-01	3.08E-01	2.36E-01	1.69E-01
1	4.91E-01	4.54E-01	4.24E-01	4.28E-01	4.21E-01	3.27E-01	2.36E-01
0.8	5.62E-01	5.22E-01	4.89E-01	4.94E-01	4.85E-01	3.80E-01	2.76E-01
0.6	6.51E-01	6.07E-01	5.71E-01	5.76E-01	5.66E-01	4.49E-01	3.29E-01
0.4	7.61E-01	7.13E-01	6.75E-01	6.80E-01	6.69E-01	5.39E-01	4.02E-01
0.3	8.26E-01	7.77E-01	7.37E-01	7.43E-01	7.31E-01	5.96E-01	4.50E-01
0.2	8.99E-01	8.49E-01	8.08E-01	8.14E-01	8.02E-01	6.62E-01	5.07E-01
0.1	9.82E-01	9.31E-01	8.89E-01	8.96E-01	8.83E-01	7.39E-01	5.78E-01
0	1.08E+00	1.03E+00	9.83E-01	9.89E-01	9.76E-01	8.31E-01	6.67E-01

The heading above the numeric columns reads "t / tau0".

The numerical solutions presented above are valid all the way up to the present time (z=0), since proper account is taken of all three terms in (3.9). However, they still suffer from the inaccuracy caused by neglecting pressure in the early universe.

Numerical Integration of Pressureless FRW Friedmann Equation: Time for a Given Red-Shift, z

Numerical Integration of Pressureless FRW Friedmann Equation: Time for a Given Red-Shift, z
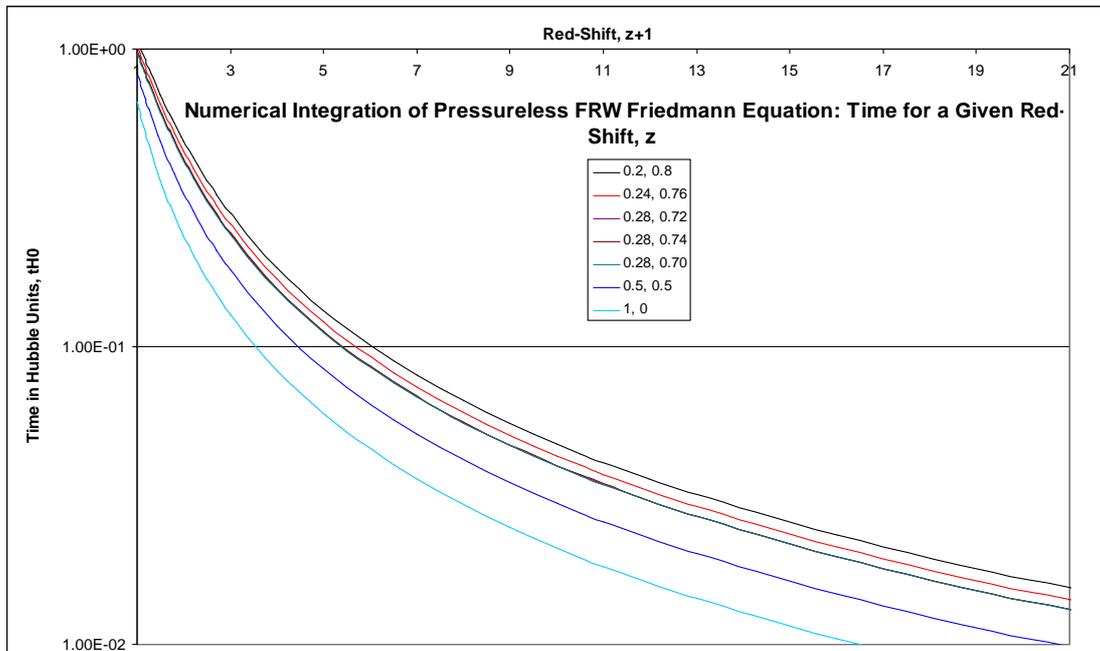
The key to the above graphs gives $\Omega_0, \Omega_{\Lambda 0}$, and Ω_{R0} follows from $\Omega + \Omega_\Lambda + \Omega_R = 1$. The best current values for the cosmological parameters are, roughly, $\Omega_0 \sim 0.24$ to 0.28; $\Omega_{\Lambda 0} \sim 0.72$ to 0.76; Ω_{R0} consistent with zero and probably lying between -0.02 and +0.02. The cases which consider non-zer0 values for Ω_{R0}, namely +/-0.02, are indistinguishable in the above graph. The difference between $\Omega_0 = 0.24$ and 0.28 is discernable but small. The graph for $\Omega_0 = 1$ was generated by the

integration procedure and was found to be identical to Equ.(3.11). The value of R(t) follows, of course, from $R(t) = R_0 / (1 + z)$. For $\Omega_0 = 0.28$, $\Omega_{\Lambda 0} = 0.72$ we find,
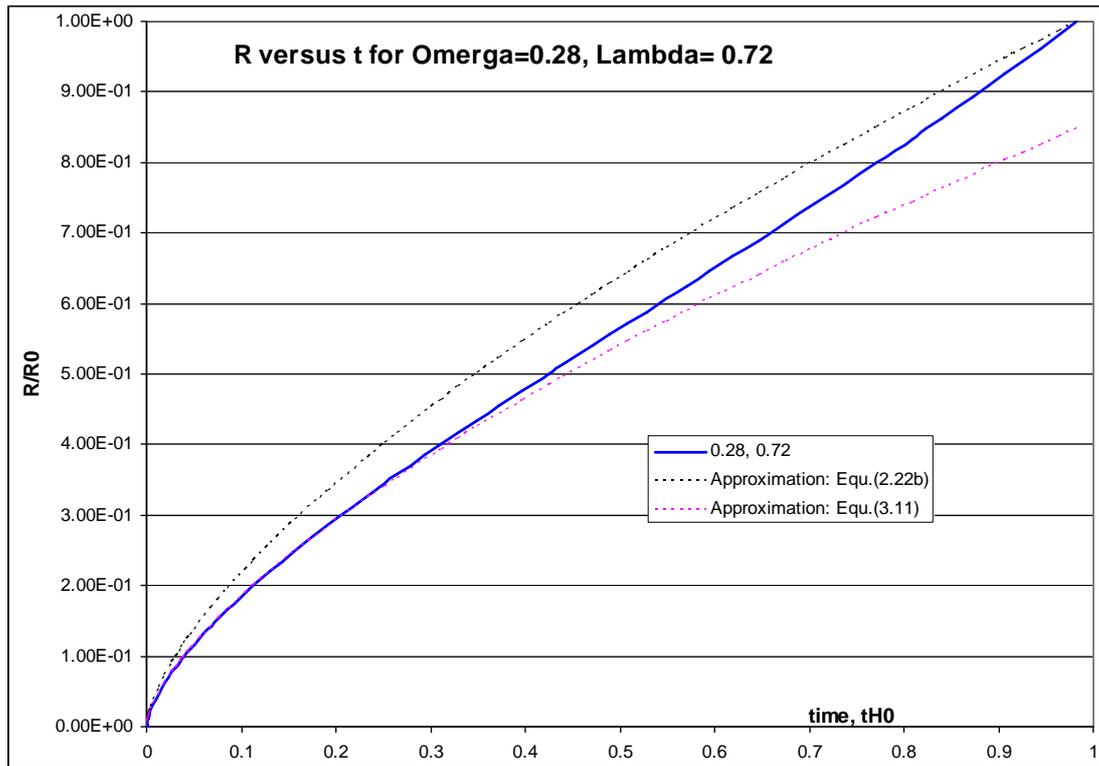
Note that the accelerating expansion rate of the universe due to the positive value for $\Omega_{\Lambda 0}$ is easily discernable in the above graph for times beyond about $0.3/H_0$, where the numerical integration diverges from the approximate result of Equ.(3.11). The approximation of Equ.(2.22/2.22b) is bound to be right at the present time, but is in error at all earlier times.
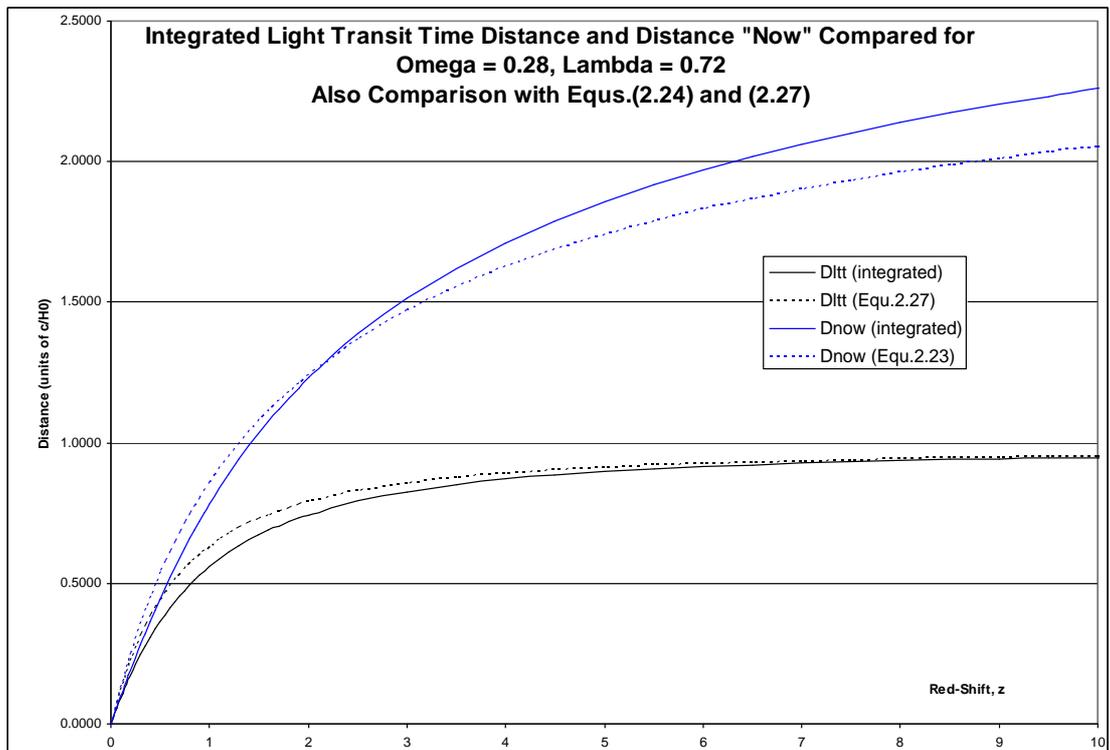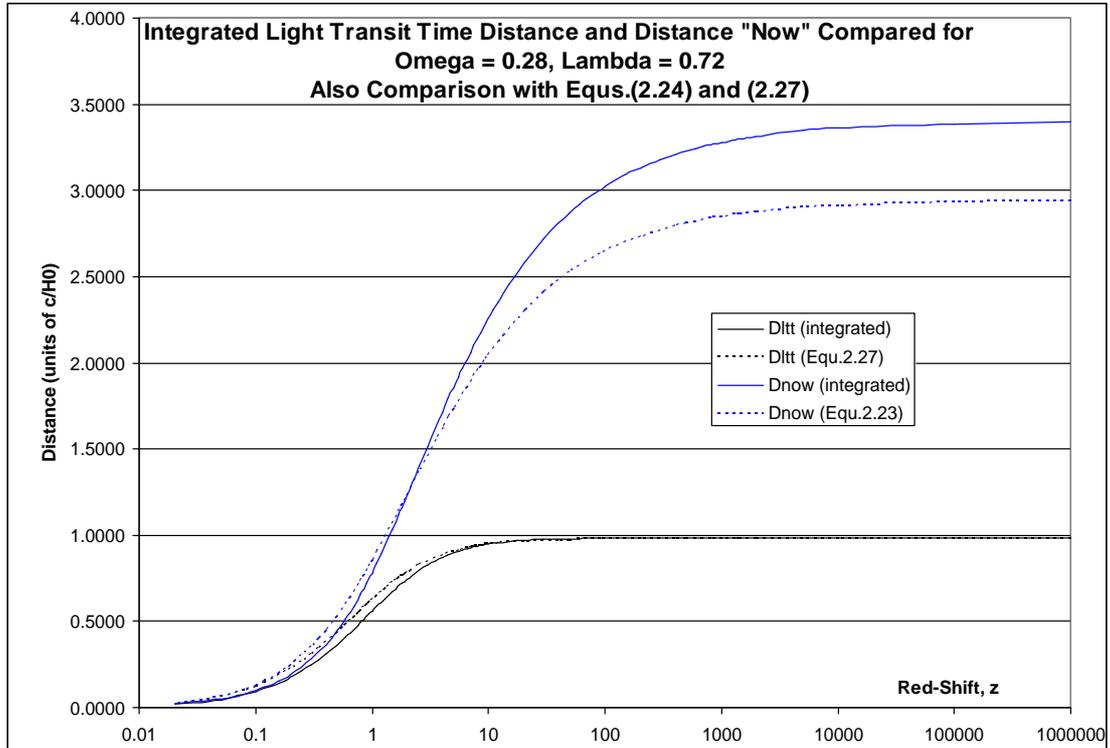
Numerical solutions for the distances D_{ltt} and D_{now} follow…

Numerical Solutions for the Distances D_{ltt} and D_{now} ($\Omega_0 = 0.28; \Omega_{\Lambda 0} = 0.76$)

z	t / tau0	D_{ltt}		D_{now}	
		numerical	Equ.(2.27)	Numerical	Equ.(2.23)
1000000	1.12E-09	0.9828	0.9828	3.3950	2.9454
100000	3.96E-08	0.9828	0.9828	3.3868	2.9390
10000	1.26E-06	0.9828	0.9828	3.3610	2.9189
2000	1.41E-05	0.9828	0.9828	3.3143	2.8824
1400	2.4E-05	0.9828	0.9828	3.2978	2.8696
1000	3.98E-05	0.9827	0.9827	3.2793	2.8551
800	5.56E-05	0.9827	0.9827	3.2653	2.8442
600	8.55E-05	0.9827	0.9827	3.2446	2.8281
400	0.000157	0.9826	0.9827	3.2101	2.8011
200	0.000442	0.9823	0.9824	3.1322	2.7404
150	0.000679	0.9821	0.9822	3.0912	2.7084
100	0.001241	0.9815	0.9818	3.0227	2.6550
80	0.001728	0.9810	0.9814	2.9789	2.6207
60	0.002644	0.9801	0.9807	2.9149	2.5708
40	0.004799	0.9780	0.9790	2.8085	2.4879
20	0.013091	0.9697	0.9726	2.5740	2.3050
15	0.019684	0.9631	0.9674	2.4539	2.2113
12	0.026874	0.9559	0.9618	2.3506	2.1306
11	0.0303	0.9525	0.9591	2.3078	2.0972
10	0.034522	0.9483	0.9558	2.2594	2.0594
9	0.039824	0.9430	0.9517	2.2038	2.0160
8	0.046635	0.9361	0.9464	2.1392	1.9656
7	0.055633	0.9271	0.9393	2.0630	1.9059
6	0.067942	0.9148	0.9297	1.9710	1.8340
5	0.085555	0.8972	0.9159	1.8571	1.7447
4	0.112304	0.8705	0.8949	1.7110	1.6298
3	0.156449	0.8263	0.8599	1.5143	1.4742
2	0.238772	0.7440	0.7936	1.2309	1.2461
1.8	0.26391	0.7189	0.7730	1.1581	1.1864
1.6	0.293634	0.6891	0.7484	1.0779	1.1199
1.4	0.329142	0.6536	0.7185	0.9893	1.0452
1.2	0.372036	0.6107	0.6816	0.8908	0.9606
1	0.424481	0.5583	0.6353	0.7808	0.8635
0.9	0.455168	0.5276	0.6075	0.7210	0.8094
0.8	0.489426	0.4934	0.5758	0.6576	0.7508
0.7	0.527799	0.4550	0.5394	0.5905	0.6871
0.6	0.570925	0.4119	0.4972	0.5194	0.6175
0.5	0.619556	0.3632	0.4478	0.4441	0.5410
0.4	0.674576	0.3082	0.3895	0.3644	0.4565
0.3	0.737034	0.2457	0.3197	0.2801	0.3625
0.2	0.808172	0.1746	0.2352	0.1913	0.2569
0.18	0.823567	0.1592	0.2161	0.1729	0.2342
0.16	0.839382	0.1434	0.1962	0.1544	0.2109
0.14	0.855629	0.1271	0.1754	0.1358	0.1870
0.12	0.872324	0.1105	0.1536	0.1169	0.1624
0.1	0.889481	0.0933	0.1309	0.0979	0.1372
0.08	0.907116	0.0757	0.1071	0.0786	0.1113
0.06	0.925246	0.0575	0.0823	0.0592	0.0847
0.04	0.943888	0.0389	0.0562	0.0397	0.0573
0.02	0.963059	0.0197	0.0288	0.0199	0.0290
0	0.982778	0.0000	0.0000	0.0000	0.0000

The above graphs show the distance measures D_{ltt} and D_{now} for a source which has the red-shift, z, plotted as the abscissa. Note that for the case $\Omega_0 = 1; \Omega_{\Lambda 0} = 0$ it was confirmed that the numerical integration exactly reproduced the analytic formulae given in Equs.(2.24, 2.27).

Integrated Light Transit Time Distance and Distance Now Compared for Omega = 0.28 and Lambda = 0.72: Plotted Against Time When Signal Was Emitted

Integrated Light Transit Time Distance and Distance Now Compared for Omega = 0.28 and Lambda = 0.72: Plotted Against Time When Signal Was Emitted
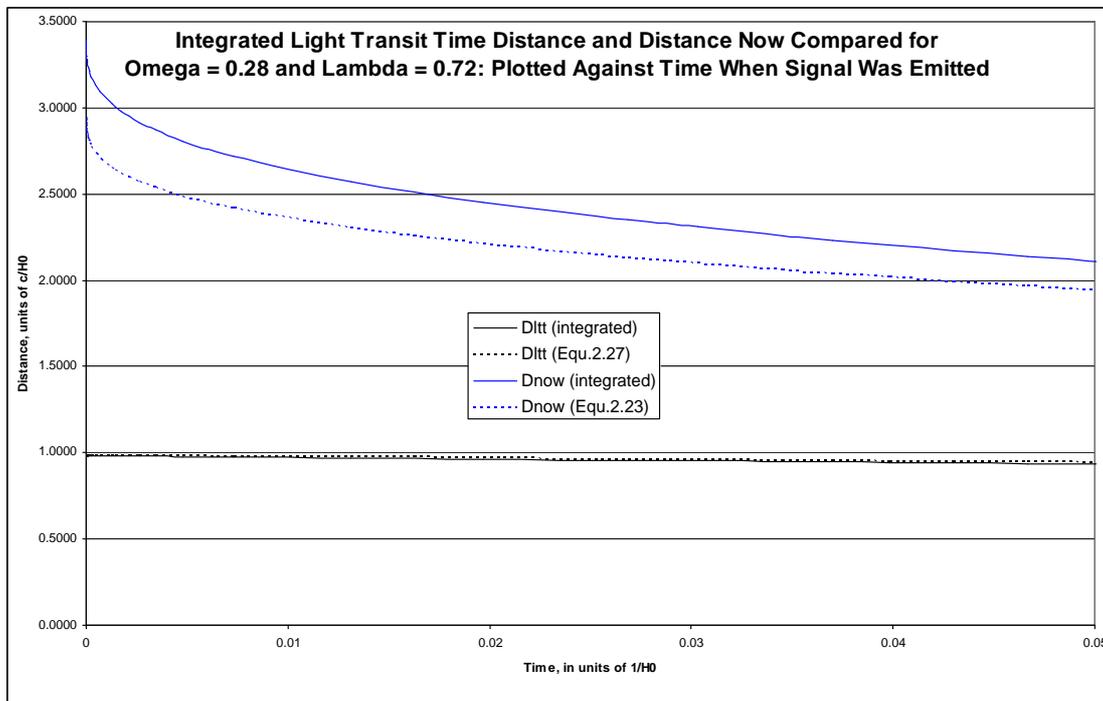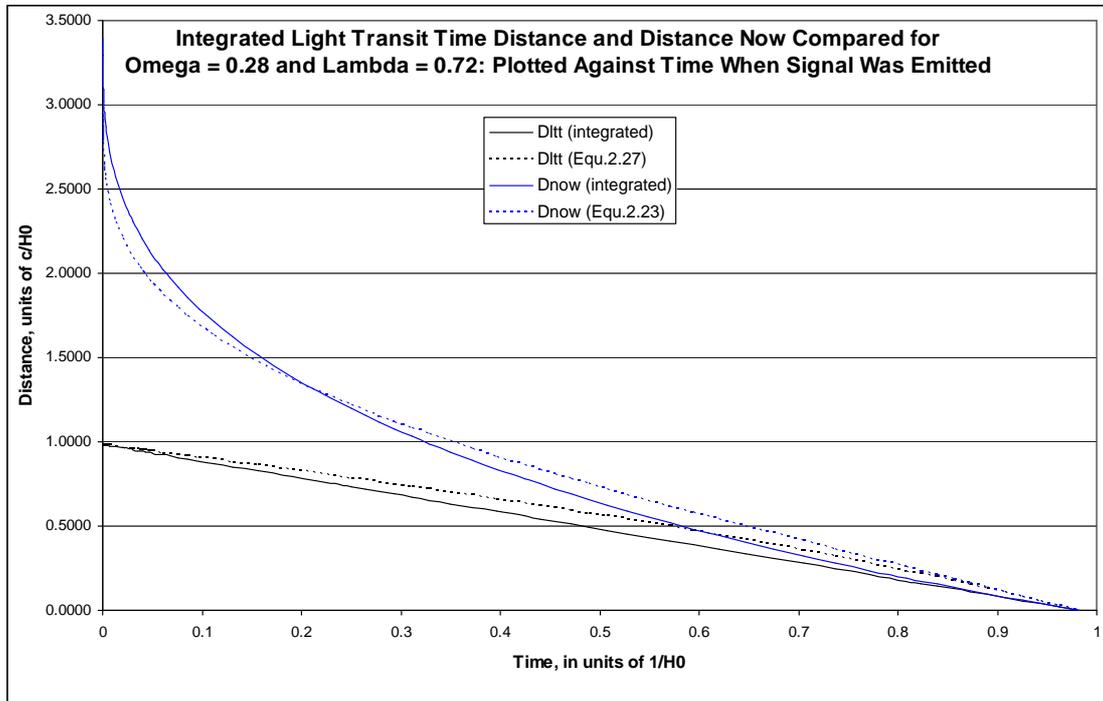
The above graphs give the distance measures of a source which emits the received signal at cosmic time t. Thus, when t/τ_0 is small, the light-transit-time distance is necessarily close to $c\tau_0$. However, the physical distance to the horizon is given by the distance measure "D_{now}" for t = 0 (or equivalently for a divergent red-shift). Hence, the horizon is at a distance of $3.40c\tau_0$ in our universe (assuming $\Omega_0 = 0.28$ and $\Omega_{\Lambda 0} = 0.76$). Note that, with these parameters, $t_{now} = 0.983\tau_0$ so that the horizon is at a physical distance $D_{now} = 3.40c\tau_0 = \underline{\textbf{3.46ct}_{\underline{\textbf{now}}}}$.

4. Hubble's Law

Hubble's Law is formulated in terms of the distance measure D_{now}. The velocity in question is the rate of change of this quantity, i.e. how rapidly the two co-moving points (galaxy clusters) appear to be moving apart <u>now</u>. We have therefore,

$$D_{now} = R(t_{now})\Delta L_A \quad \text{and} \quad V_{now} = \frac{dD_{now}}{dt} = \dot{R}(t_{now})\Delta L_A \quad (6.?)$$

Hence, the angular separation cancels between the distance and velocity measures, giving,

$$V_{now} = H(t)D_{now} \quad \text{where} \quad H(t) \equiv \frac{\dot{R}(t_{now})}{R(t_{now})} \quad (6.?)$$

The essence of the Hubble relation is that the parameter H does not depend upon spatial position, only upon time. The origin of this is clear from the cosmological geometry – provided the right distance and velocity measures are used! Note that these 'correct measures' are such that the velocity may exceed c (and is 3c at the boundary of the observable universe, for flat space-time at least).