Chapter 2b – The Flatness Problem

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1. Introduction

In Chapter 2 we saw that the universe inevitably starts extremely close to the critical density. Thereafter the density would be expected to diverge away from the critical value, depending upon the net energy, E. The net energy, E, may be parameterised by the value of Ω at some convenient early time (say, at 1 second). The value of Ω at such an early time is inevitably very close to unity, but quickly diverges away from unity. Hence, to give rise to a value for Ω at the present epoch which is within even a few orders of magnitude of unity requires a value of Ω at 1 second which is exceedingly close to unity – to an accuracy of a great many decimals places. This is one of the famous instances of fine tuning. Specifically we shall see that, for Ω to be compatible with observations (lying, say, between 0.1 and 2[1]) it is necessary that its value at 1 second lie within +/-10^{-15} of unity. Why Ω at early times should be so closely fine tuned to unity is known as the "Flatness Problem".

In this Chapter we illustrate by specific calculations the sensitivity of Ω to its 'boundary condition' value at early times.

2. Evaluating Ω As A Function of Time

Chapter 2 has provided the means of calculating Ω, since Equ.(2.3.14) together with Equ.(2.2.7) are:-

Matter Dominated Era

$$\Omega = 1 - \frac{E}{E + \frac{GM^2}{R}}, \quad \text{and} \quad \frac{R}{R_0} = \left(\frac{t}{t_0}\right)^{2/3} \qquad (2b.2.1)$$

Radiation Dominated Era

$$\Omega = 1 - \frac{E}{E + \frac{GM\lambda_R}{R^2}}, \quad \text{and} \quad \frac{R}{R_0} = \left(\frac{t}{t_0}\right)^{1/2} \qquad (2b.2.2)$$

(Note that the radiation era constant λ_R can be written as M'R, where M' is the total mass, including radiation energy mass equivalent. In the radiation era, M' is dominated by the radiation, whose energy is falling as 1/R, hence the constancy of M'R).

In Equs.(2b.2.1,2), the subscript $_0$ denotes some convenient time which is used, in effect, to determine the coefficient of proportionality. We may use the second equation of each pair to reformulate the first, using the following dimensionless parameters,

[1] This is grossly conservative since the value of Ω, including dark energy, is currently believed to lie within 2% of unity.

$$\xi = \frac{E}{GM^2 / R_0} \qquad \text{and} \qquad \tilde{\xi} = \frac{E}{GM\lambda_R / R_0^2} \qquad (2b.2.3)$$

giving,

Matter Dominated Era

$$\Omega = 1 - \frac{\xi}{\xi + (R_0 / R)} = 1 - \frac{\xi}{\xi + (t_0 / t)^{2/3}} \qquad (2b.2.5)$$

Radiation Dominated Era

$$\Omega = 1 - \frac{\tilde{\xi}}{\tilde{\xi} + (R_0 / R)^2} = 1 - \frac{\tilde{\xi}}{\tilde{\xi} + (t_0 / t)} \qquad (2b.2.6)$$

Note that ξ and $\tilde{\xi}$ are constants, so that Equs.(2b.2.5,6) give the time variation of Ω explicitly.

3. 'Initial' Value For Ω To Give Present Observed Value

For the sake of this illustration we assume that observational evidence guarantees that the present value of Ω lies in the range 0.1 to 2.0[2]. Our question is, "what must the value of Ω have been at 1 second after the Big Bang to be consistent with the present range of values?" We shall take the current age of the universe to be 13.7 billion years ($t_{now} = 4.32 \times 10^{17}$ sec). We shall also need the time at which the radiation era gives way to the matter dominated era, since the time dependence of Ω is different for the two. The radiation era will be taken to end at 500,000 years ($t_{EQ} = 1.58 \times 10^{13}$ sec) for the sake of this illustration.

(a) Ω_{now} = 0.1

Period t_{EQ} to t_{now}: This period is matter dominated. Use t_{now} as a convenient definition of t_0. Thus, Equ.(2b.2.5) gives,

$$\Omega_{now} = 0.1 = 1 - \frac{\xi}{\xi + (1)^{2/3}} = \frac{1}{1+\xi} \quad \text{hence} \quad \xi = 9$$

Hence,
$$\Omega_{EQ} = 1 - \frac{9}{9 + \left(4.32x10^{17} / 1.58x10^{13}\right)^{2/3}} = 1 - 0.009807$$

Period 1 sec to t_{EQ}: This period is radiation dominated. Use t_{EQ} as a convenient definition of t_0. Thus, Equ.(2b.2.6) gives,

$$\Omega_{EQ} = 1 - 0.009807 = 1 - \frac{\tilde{\xi}}{\tilde{\xi} + (1)} = \frac{1}{1+\tilde{\xi}} \quad \text{hence} \quad \tilde{\xi} = 0.009904$$

[2] This is a gross understatement. Observational evidence favours $\Omega \approx 1$ to within about 1%.

Hence, $$\Omega(1\sec) = 1 - \frac{0.009904}{0.009904 + \left(1.577x10^{13}/1\right)} = 1 - 6.3x10^{-16}$$

Thus, to produce a value Ω_{now} = 0.1 we require a value of Ω at 1 second which differs from unity by less than 10^{-15}.

(b) Ω_{now} = 2.0
Period t_{EQ} to t_{now}: This period is matter dominated. Use t_{now} as a convenient definition of t_0. Thus, Equ.(2b.2.5) gives,

$$\Omega_{now} = 2.0 = 1 - \frac{\xi}{\xi + (1)^{2/3}} = \frac{1}{1+\xi} \quad \text{hence} \quad \xi = \text{-0.5}$$

Hence, $$\Omega_{EQ} = 1 - \frac{-0.5}{-0.5 + \left(4.32x10^{17}/1.58x10^{13}\right)^{2/3}} = 1 + 0.000551$$

Period 1 sec to t_{EQ}: This period is radiation dominated. Use t_{EQ} as a convenient definition of t_0. Thus, Equ.(2b.2.6) gives,

$$\Omega_{EQ} = 1 + 0.000551 = 1 - \frac{\tilde{\xi}}{\tilde{\xi} + (1)} = \frac{1}{1+\tilde{\xi}} \quad \text{hence} \quad \tilde{\xi} = -0.000551$$

Hence, $$\Omega(1\sec) = 1 - \frac{-0.000551}{-0.000551 + \left(1.577x10^{13}/1\right)} = 1 + 3.5x10^{-17}$$

Thus, to produce a value Ω_{now} = 2.0 we require a value of Ω at 1 second which differs from unity by less than 10^{-16}.

Even if we widen the possible range of Ω today, by a factor of 1000, an extremely high degree of fine tuning of the value of Ω to unity is required at 1 second. Thus:-

(c) Ω_{now} = 0.001
Period t_{EQ} to t_{now}: This period is matter dominated. Use t_{now} as a convenient definition of t_0. Thus, Equ.(2b.2.5) gives,

$$\Omega_{now} = 0.001 = 1 - \frac{\xi}{\xi + (1)^{2/3}} = \frac{1}{1+\xi} \quad \text{hence} \quad \xi = 999$$

Hence, $$\Omega_{EQ} = 1 - \frac{999}{999 + \left(4.32x10^{17}/1.58x10^{13}\right)^{2/3}} = 1 - 0.524$$

Period 1 sec to t_{EQ}: This period is radiation dominated. Use t_{EQ} as a convenient definition of t_0. Thus, Equ.(2b.2.6) gives,

$$\Omega_{EQ} = 1 - 0.524 = 1 - \frac{\tilde{\xi}}{\tilde{\xi} + (1)} = \frac{1}{1+\tilde{\xi}} \quad \text{hence} \quad \tilde{\xi} = 1.1007$$

Hence, $$\Omega(1\sec) = 1 - \frac{1.1007}{1.1007 + \left(1.577x10^{13}/1\right)} = 1 - 7.0x10^{-14}$$

Thus, to produce a value $\Omega_{now} = 0.001$ we require a value of Ω at 1 second which differs from unity by less than 10^{-13}.

(d) Ω_{now} = 1000

Period t_{EQ} to t_{now}: This period is matter dominated. Use t_{now} as a convenient definition of t_0. Thus, Equ.(2b.2.5) gives,

$$\Omega_{now} = 1000 = 1 - \frac{\xi}{\xi + (1)^{2/3}} = \frac{1}{1+\xi} \quad \text{hence} \quad \xi = -0.999$$

Hence, $$\Omega_{EQ} = 1 - \frac{-0.999}{-0.999 + \left(4.32x10^{17}/1.58x10^{13}\right)^{2/3}} = 1 + 0.001102$$

Period 1 sec to t_{EQ}: This period is radiation dominated. Use t_{EQ} as a convenient definition of t_0. Thus, Equ.(2b.2.6) gives,

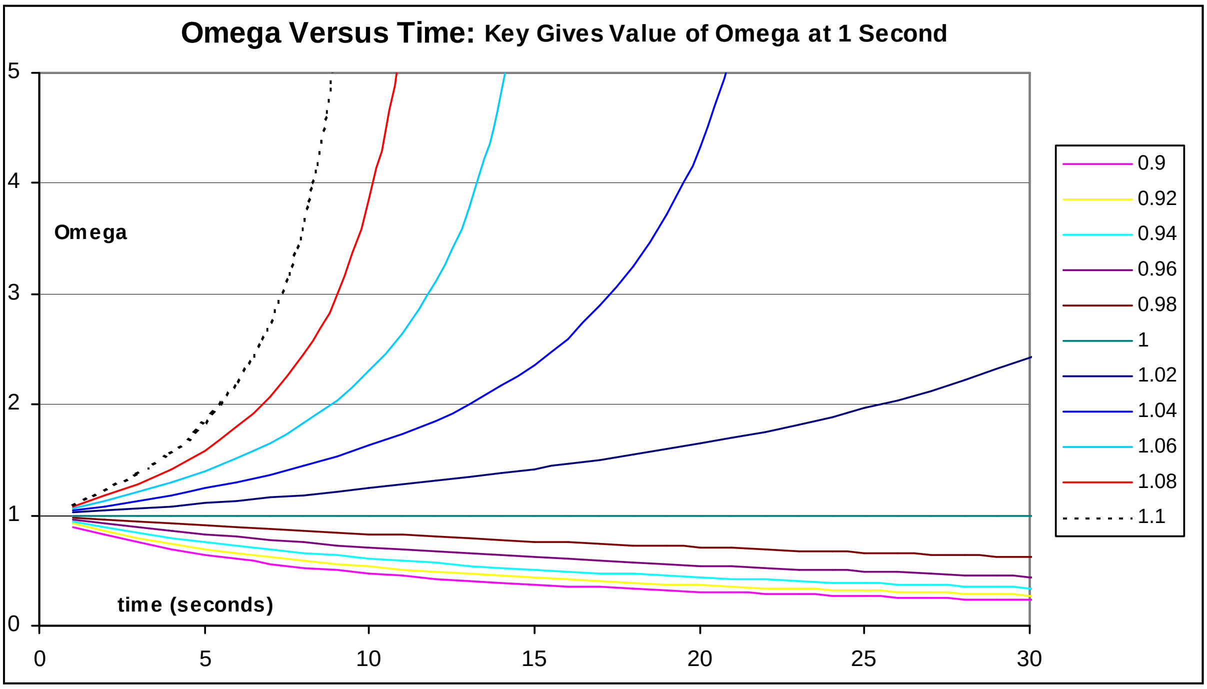
$$\Omega_{EQ} = 1 + 0.001102 = 1 - \frac{\tilde{\xi}}{\tilde{\xi} + (1)} = \frac{1}{1+\tilde{\xi}} \quad \text{hence} \quad \tilde{\xi} = -0.001101$$

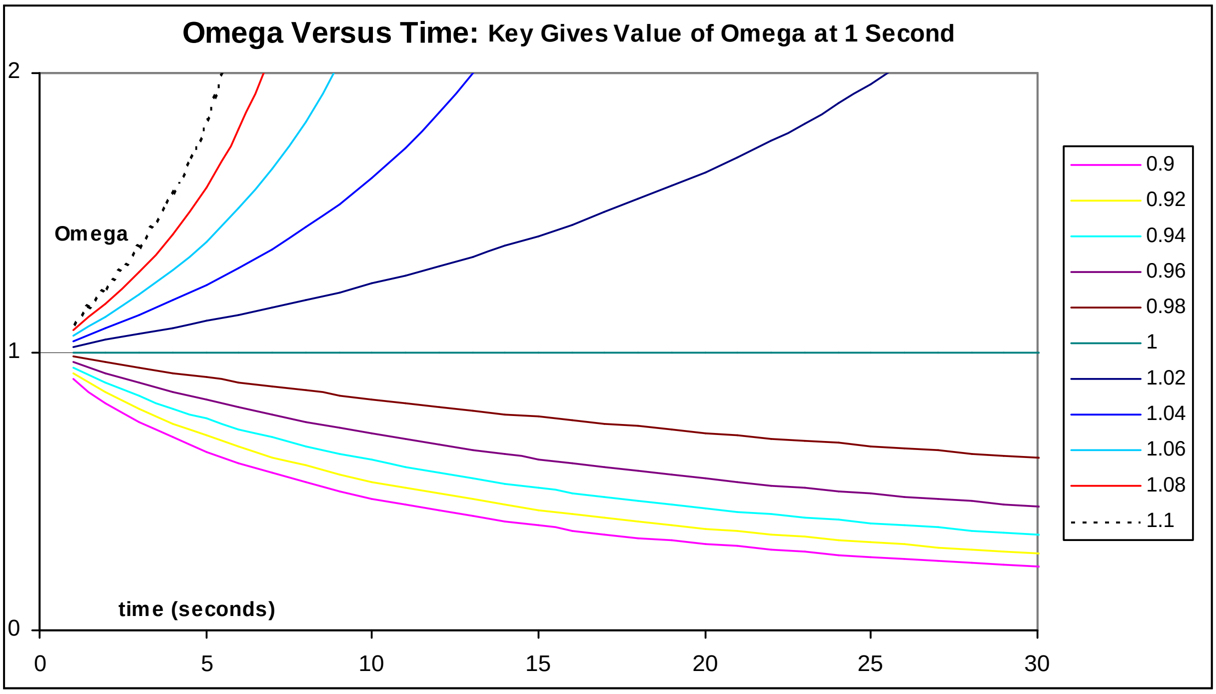
Hence, $$\Omega(1\sec) = 1 - \frac{-0.001101}{-0.001101 + \left(1.577x10^{13}/1\right)} = 1 + 7.0x10^{-17}$$

Thus, to produce a value $\Omega_{now} = 1000$ we require a value of Ω at 1 second which differs from unity by less than 10^{-16}. Note that this Ω at 1 second differs from unity only by twice the amount that gives rise to $\Omega_{now} = 2$ (see case b, above).

4. Graphs of Ω Versus Time

The graphs which follow plot Ω against time for a range of values of Ω at 1 second. We shall only be concerned with early times so these results are based upon radiation domination. The period extends only to 30 seconds but is sufficient to illustrate the very rapid rate of divergence away from a value for Ω of unity. To still have a value of near unity at the present epoch seems rather like balancing a pencil on its tip – only more so.


Omega Versus Time: Key Gives Value of Omega at 1 Second
Omega
time (seconds)
0
1
2
3
4
5
0
5
10
15
20
25
30
0.9
0.92
0.94
0.96
0.98
1
1.02
1.04
1.06
1.08
1.1


Omega Versus Time: Key Gives Value of Omega at 1 Second
Omega
time (seconds)
0
1
2
0
5
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15
20
25
30
0.9
0.92
0.94
0.96
0.98
1
1.02
1.04
1.06
1.08
1.1

5. Conclusion

The Flatness Problem has been illustrated by the examples in Sections 3 and 4. Note that Equs.(2b.2.1,2) can be written,

$$\Omega = 1 - \frac{E}{E_{KE}} \qquad (2b.5.1)$$

where the term in the denominator is the kinetic energy. The net energy, E, is the sum of E_{KE} and the (negative) gravitational potential energy. Hence, Ω equals unity if E = 0, in other words if E_{KE} cancels with the gravitational potential energy. Consequently, the condition for flatness,

$$\Omega \approx 1 \qquad (2b.5.2)$$

can also be written as,

$$E \approx 0 \qquad (2b.5.3)$$

This may be the most stunning thing in cosmology – the universe adds up to nothing. To quote Alan Guth, "the universe is the ultimate free lunch".

Because the net energy is zero, it has been suggested that the universe could have arisen as a quantum fluctuation. Ordinarily a quantum fluctuation does not last very long. Specifically, a quantum fluctuation of energy E would be expected to last for a time of about $\hbar / E$. If the quantum fluctuation that is the universe has lasted, so far, for 13.7 billion years then its energy must be very small indeed – in fact around 10^{-52} Joules. Hence, from this perspective, the flatness of the universe becomes a prerequisite for its longevity. Appealing though this may be, there are theoretical difficulties with this idea. For example, in an infinite universe the quantum fluctuation would have to occur simultaneously at all points in an infinite space.

Currently the most widely accepted explanation of the very high degree of flatness of the universe is inflation theory.