Appendix I
Type II Supernovae: Details of Carr & Rees Coincidence

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I.1 The Argument of Carr & Rees (1979)

In moving through a dense star, neutrinos can react with the nucleons, either free nucleons or those within nuclei, via weak nuclear processes such as $\nu_e + n \rightarrow e^- + p^+$. If a Type II supernova is in progress then we can imagine the material of the mantle expanding rapidly outwards. This expansion will have some characteristic timescale, called the dynamical timescale, defined by the initial speed and the resisting pull against gravity. Carr and Rees argued that the neutrino reaction time must be of the same order as the dynamical timescale. The reasoning is that, firstly, the neutrinos mostly escape the star's core, rather than being trapped within it. Consequently, the reaction time cannot be too short compared with the dynamical timescale. On the other hand, if the explosion is *caused* by the neutrinos, then it is necessary that some interactions occur before they escape. So the reaction time cannot be too long compared with the dynamical timescale either. Hence, the two timescales must be of the same order. We proceed to estimate each of these timescales.

The Dynamical Timescale

Imagine for simplicity a single particle of mass m and initial radial speed v_0, moving outward from a central gravitating point mass M, starting from a radius r_0. Newtonian mechanics readily gives the equation to solve to find r at any time,

$$\frac{dr}{dt} = \left[v_0^2 - 2GM\left(\frac{1}{r_0} - \frac{1}{r} \right) \right]^{1/2} \qquad \text{(I.1)}$$

To make the maths simple we assume that the original K.E. is equal and opposite to the original P.E. This of course is the condition that the particle has zero total energy, i.e. it can just escape to infinity (and v_0 is the escape velocity at radius r_0). Note that this condition is definitely *not* true on average for a bound system (obviously), the virial theorem stating that K.E. = -P.E./2. However, we are dealing with the mantle of a supernova, and this is not bound since it is in the process of being blown off. With our simplifying assumption, together with the additional assumption that $r_0 << r$, we get,

$$t_{dynamic} \approx \frac{\sqrt{2}}{3} \cdot \frac{r^{3/2}}{\sqrt{GM}} \qquad \text{(I.2)}$$

Imagine that the mass M comprises N nucleons (including those within nuclei) and that the whole mass, assumed uniform, is expanding from r_0 to r. The mass may be written in terms of the number density of nucleons as,

$$M = M_p \rho_N \frac{4}{3} \pi r^3 \qquad \text{(I.3)}$$

Substitution into (I.2) gives,

$$t_{dynamic} \approx \frac{\sqrt{2}}{6} \cdot \frac{1}{\sqrt{GM_p \rho_N}} \tag{I.4}$$

The numerical coefficient is hardly justified given that we have glossed over the difference between a point gravitating mass and the uniform density case. The dynamical timescale depends only upon the nucleon number density (including those in nuclei), apart from universal constants.

<u>The Neutrino-Nucleon Reaction Timescale</u>

Reactions like $\nu_e + n \to e^- + p^+$ have a cross section given roughly by,

$$\sigma = \frac{8}{\pi} G_F'^2 E_\nu^2 \tag{I.5}$$

(Although the 8 might be more accurately replaced by 5.8 - Check). Reactions like this will also occur within nuclei, so the whole nucleon density counts – though the cross sections may be a bit different.

In addition, and possibly of crucial importance, is that neutral weak reactions will also be occurring. These are weak reactions mediated by the neutral Z boson, and consequently do not involve an exchange of charge. Obvious examples are elastic reactions like $\nu + n \to \nu + n$ and $\nu + p \to \nu + p$. For these reactions any of the three neutrino species, or their three antineutrinos, can be involved. Hence the mu and tau neutrinos, and their antineutrinos, can contribute – some claim crucially. We shall assume that (I.5) is sufficient to cover each of the contributing neutrino reactions to sufficient accuracy.

The number of reactions per second per neutrino is $\sigma\rho_N c$. Consequently the reaction time is,

$$\tau_\nu = 1/\sigma\rho_N c = \left[\frac{8}{\pi} G_F'^2 E_\nu^2 \rho_N c\right]^{-1} \tag{I.6}$$

and the mean free path of a neutrino is,

$$\lambda_\nu = c\tau_\nu = 1/\sigma\rho_N \tag{I.7}$$

The Carr & Rees estimate equates (I.4) with (I.6), which gives,

$$GM_p \sim 0.36 G_F'^4 E_\nu^4 \rho_N c^2 \tag{I.8}$$

In the case of the Type II supernova core, the density is roughly the nuclear density, which can be written,

$$\rho_N^{nuclear} \approx \frac{1}{4}\left(\frac{M_\pi c}{\hbar}\right)^3 \qquad (I.9)$$

where M_π is the pion mass (135-140 MeV) (which gives a nuclear density of $M_p \rho_N^{nuclear} \sim 2 \times 10^{17}$ kg/m^3). Using (I.9), and substituting the dimensionless Fermi constant using $G_F = \alpha_w^p \hbar^3 / M_p^2 c$, then (I.8) gives,

$$\alpha_w^p \sim 1.83 \alpha_G^{1/4} \left(\frac{M_p}{M_\pi}\right)^{3/4} \cdot \frac{M_p c^2}{E_\nu} \qquad (I.10)$$

where $\alpha_G = \frac{GM_p^2}{\hbar c} = 5.88x10^{-39}$. Carr & Rees went on to argue that the relevant neutrino energy was $E_\nu \sim m_e c^2$ (apparently based on neutrinos originating from electron/positron annihilation). In which case (I.10) becomes,

$$\alpha_w^p \sim 1.83 \alpha_G^{1/4} \left(\frac{M_p}{M_\pi}\right)^{3/4} \cdot \frac{M_p}{m_e} \qquad (I.11)$$

Carr & Rees used the nucleon mass to estimate the nuclear density, i.e. in (I.9), which is quite a bad approximation. Hence their version of (I.11) did not include the nucleon:pion mass ratio term.

Evaluating the RHS of (I.11) gives 0.4 x 10^{-5}. This is indeed a very creditable order of magnitude estimate for the dimensionless Fermi constant, which is actually 1.03 x 10^{-5}.

Equ.(I.11) is strikingly similar to the result of constraining the primordial universe to contain roughly similar amounts of hydrogen and helium, i.e. equation (6.3.11), $\alpha_w^p \sim 0.887 \alpha_G^{1/4} \left(\frac{M_p}{\Delta M}\right)^{3/2}$. Requiring both (6.3.11) and (I.11) to hold suggests a 'coincidence' between masses, i.e.,

$$0.887 \left(\frac{M_p}{\Delta M}\right)^{3/2} \approx 1.83 \left(\frac{M_p}{M_\pi}\right)^{3/4} \cdot \frac{M_p}{m_e} \qquad (I.12)$$

i.e.,

$$M_\pi = 2.63 \frac{M_p^{1/3} \Delta M^2}{m_e^{4/3}} \qquad (I.13)$$

The RHS of (I.13) evaluates to 105 MeV, which is a very creditable estimate of the pion mass (135 to 140 MeV), given that the contributing masses range from 0.51 MeV to 983.3 MeV.

However, having seemed to heap praise on this argument, we now proceed to rather pour cold water over it…

I.2 Critique of the Carr & Rees Type II Supernova Anthropic Constraint
There are two points at which the argument of Section I.1 can be criticised quite harshly. The first is that, in deriving the dynamical timescale, we made an arbitrary assumption regarding the initial energy (speed) of the nucleons, namely that it corresponded to K.E. + P.E. ~ 0. The second is that, following Carr & Rees, we have assumed a neutrino energy of about $E_\nu \sim m_e c^2$. We examine these assumptions more closely below.

How much energy is there in a Type II supernova? Nuclear fusion reactions, at least the exothermic ones, have already ceased. The source of the Type II supernova's energy is therefore gravitational collapse. We have seen that the mass of the core must be at least $1.4\,M_\Theta$ - let's call it $1.5\,M_\Theta$ for sake of argument. Also, the density of the core is nuclear density, i.e. ~2 x 10^{17} kg/m^3. From these we deduce that the core must be of radius 15km or so. The density of the core of an evolved star of this mass prior to going supernova is around 3 x 10^{10} kg/m^3, and hence the initial radius would have been about 2880 km. Hence we can ignore the initial potential energy and approximate the energy released as (minus) the final potential energy, i.e.,

$$\text{Supernova Energy } \approx \frac{3}{5}\cdot\frac{GM^2}{R} \qquad \text{(I.14)}$$

where R ~ 15 km and M ~ $1.5\,M_\Theta$ ~ 3 x 10^{30} kg. Hence the energy is a prodigious 2.4 x 10^{46} J = 1.5 x 10^{59} MeV.

The number of nucleons in the core is M/M_p = 1.8 x 10^{57}. Hence the energy per nucleon is ~84 MeV (i.e. almost 10% of the rest mass). This is a heck of a lot of energy. If it were thermalised the temperature would be 6 x 10^{11} K. This energy corresponds to an initial nucleon speed of 1.3 x 10^8 m/s, or ~40% of light speed. However, this model *does* confirm that the nucleon K.E. + P.E. is virtually zero just prior to the explosive expansion, as we assumed in deriving the dynamical timescale. There is, however, some confusion about the distinction between the core and the mantle.

There is a more serious issue as regards the energy of the neutrinos. The nucleon energy must be transferred to the neutrinos if it is to escape from the star. If we assume that this is accomplished in a single nucleon-neutrino interaction then the energy of 84 MeV transfers to the neutrino. Of course, we could transfer the energy in several interactions – transferring smaller amounts at a time – but the whole 84 MeV must be transferred eventually. The average neutrino energy thus depends simply on how many neutrinos there are compared with nucleons. The standard assumption seems to be that there are equal numbers of each type of neutrino as there are nucleons, although it is not immediately obvious why. Initially there will be equal numbers of electron neutrinos as there were protons as the reaction $e + p \to n + \nu_e$ turns all the protons into neutrons. The other neutrinos are made by pair production. However, assuming there are indeed the same number of each neutrino type as nucleons, and since there are 6 species of neutrino, including antineutrinos, the average neutrino energy will be 84 / 6 = 14 MeV. This is why the existence of 3 neutrino species may be crucial. This is far larger than the energy of $m_e c^2$ ~0.5 MeV assumed in the Carr & Rees argument.

The 1987a supernova has confirmed that the emitted neutrinos do indeed have large energies, the 'prompt' neutrinos being in the range 5 MeV to 40 MeV, with quite a few at the high end of this range. This seems to confirm the above picture, at least crudely. Incidentally, the total number of neutrinos detected was 20. This suggests a flux of neutrinos at Earth distance which aligns well with the expected total production of neutrinos.

This greater energy estimate has a dramatic effect on the mean free path of the nucleons. Using (I.7) with a neutrino energy of 14 MeV, and assuming nuclear densities, gives a mean free path for a neutrino in the core of ~4m. Clearly such energetic neutrinos are trapped in the core. This does appear to be the modern view of things, with the neutrinos which escape the star originating from a shell on the outer surface of the core just a few metres thick.

(Incidentally, using an energy of $m_e c^2$ ~0.5 MeV gives a mean free path in the core of 3.3 km, which is not inconsistent with a large fraction of such lower energy neutrinos escaping from the 15 km radius core).

We can now ask, "what must the density of the mantle be, during the supernova, if its thickness is to be comparable with a neutrino's mean free path assuming a realistic neutrino energy?". For a range of supposed mantle thicknesses, the Table below gives the mantle density such that the neutrino mean free path from (I.7) equals this thickness. Note that this is effectively applying the Carr & Rees approach to the mantle rather than the core, i.e., "many neutrinos escape but many neutrinos interact also".

Mantle Thickness m	**Required Mantle Density, kg/m^3**		
	E_ν = 10 MeV	**E_ν = 20 MeV**	**E_ν = 40 MeV**
10^{11}	1.2×10^7	3.1×10^6	7.7×10^5
10^{10}	1.2×10^8	3.1×10^7	7.7×10^6
10^9	1.2×10^9	3.1×10^8	7.7×10^7
10^8	1.2×10^{10}	3.1×10^9	**7.7×10^8**
10^7	1.2×10^{11}	3.1×10^{10}	7.7×10^9
10^6	1.2×10^{12}	3.1×10^{11}	7.7×10^{10}
10^5	1.2×10^{13}	3.1×10^{12}	7.7×10^{11}

In reality the density falls rapidly away from the core (in the pre-supernova star). As we proceed outwards from the layer immediately adjacent to the core, in which silicon would be burning, through the concentric shells to the outermost active shell in which hydrogen is still burning, the density drops from ~10^{10} kg./m^3 to ~10^5 kg./m^3. This last shell may be at very roughly a mass fraction of ~0.2. Outside of that there is a huge, very low density hydrogen envelope. This extends to at least ~10^{11} m. Extremely crudely, if we take an average density for the active region to be ~10^7 kg/m^3, and assume a 20 solar mass star and a mass fraction of 0.2 within the active region, a figure for the radius of the active region of ~10^8 m results. The mean free path of a 40

MeV neutrino equals 10^8 m for a density of 7.7 x 10^8 m (from the above Table). Since we have assumed a density only 1/77 of this, it suggests that only about 1% of the neutrinos would interact. This is perhaps a little low, but it is broadly consistent with what is generally found, i.e. that only a few percent of the energy of a Type II supernova ends up in the ejected remnant.

The Carr and Rees relation is, from (I.8),

$$GM_p \sim 0.36 G_F'^4 E_\nu^4 \rho_N c^2 \qquad \text{or} \qquad G_F' = \frac{1.29}{E_\nu}\left[\frac{GM_p}{\rho_N c^2}\right]^{1/4} \qquad \text{(I.15)}$$

Substituting the values suggested above for the neutrino energy (40 MeV) and the mean density of the active shell regions (10^7 kg/m^3) this gives an estimate of the dimensionless Fermi constant, $\tilde{G}_F$, of 1.7 x 10^{-5}, which is surprisingly good. The significance of this is, however, another matter.