Chapter 10 – The Constraints on Particle Masses Arising From The Stability of Nuclei and Atoms

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1. Introduction

In Chapter 9 we considered the constraints on the strength of the nuclear and electrostatic forces arising from the requirement that the nuclei of atoms should be stable. In this Chapter we shall consider what constraints apply to the masses of the nucleons and the electron for nuclei to be stable, and for solids composed of atoms to display rigidity. We shall also consider whether the stability of atoms places any further constraint on the strength of the electrostatic force.

I can recall my first exposure to the fact that the neutron is more massive than the proton. It seemed all wrong to me. I imagined the proton to be a neutron with added electric charge. Since positive potential energy, and hence mass, must be associated with the charge, surely the proton's mass should exceed that of the neutron? And yet it does not. I also remember when I first learnt that free neutrons are unstable. Again my reaction was that this seems wrong. What strange twist is this, that one of the constituents of stable matter is unstable itself? In this Chapter we see why the instability of the neutron, and $M_n > M_p$, are essential in a complex universe.

2. Stability of the Atomic Nucleus Against Electron Capture

The capture of an atomic electron by the nucleus, thereby converting a proton into a neutron, is not merely a theoretical possibility but a common mode of radioactive decay. In this universe, the mass of the neutron exceeds the combined masses of a proton plus an electron. Consequently, electron capture is possible only if the mass-energy deficit is paid for by a sufficiently large increase in the binding energy of the nucleus (see Chapter 9). Hence, the fact that $M_n > M_p + m_e$ ensures that, in this universe, only *some* nuclei are susceptible to instability by electron capture. This is because, in most cases, the nucleus would not increase its binding energy by capturing an electron – or at least not sufficiently.

However, in a universe with $M_n < M_p + m_e$ the nuclear binding energy requirement becomes dramatically relaxed. There would now be no deficit in energy, but rather energy would be released by the capture. Thus, the nuclear binding energy would not be required to increase. In fact a decrease of nuclear binding energy would be tolerable, up to a maximum of the mass-energy released, i.e., $(M_p + m_e) - M_n$. We saw in Chapter 8 that if a weak decay of a nucleus is energetically possible then it virtually always occurs. The only question is at what rate. The energy requirement in a universe with $M_n < M_p + m_e$ is so relaxed that virtually, perhaps literally, all nuclei which would otherwise have been stable are rendered unstable.

The only possibility that might save us from disaster is if the decay rates of key elements might be sufficiently slow. However, this hope is quickly dashed. A brief examination of electron capture half-lives in this universe shows that, whilst a few might be in the order of billions of years, more typically they are in the order of hours, years, or perhaps thousands of years. All these lifetimes, of less than a billion years or so, are sufficiently small compared with the timescale required for evolution that such isotopes can play no part in biochemistry. In a universe with $M_n < M_p + m_e$ very few, if any, elements would have stable isotopes of sufficient longevity to give rise to life.

We conclude that the condition $M_n > M_p + m_e$ is essential in any universe which supports chemically based life. At least, this is true so long as we assume that physics is not changed fundamentally in some way so that the conversion $p + e \rightarrow n + \nu_e$ becomes forbidden. This could be contrived, for example, in a world with no weak interactions (see Chapter 8?), or in a world in which the conservation of some novel quantum numbers prohibits such reactions.

Since $M_n > M_p + m_e$ it follows that the decay of a free neutron is inevitable in a world with weak interactions but with atoms which are stable against electron capture. We cannot have it both ways. So long as weak interactions exist, then either the masses obey $M_n < M_p + m_e$, in which case all atoms are unstable to electron capture but free neutrons are stable, or $M_n > M_p + m_e$, in which case some atoms can be stable against electron capture but free neutrons are unstable. The former would result in a universe in which the only matter was a gas of free neutrons[1].

3. Nucleon Mass Constraints Due To Nuclear Stability Against Beta Decay

We have seen in Chapter 9 that a nucleus is necessarily stable to beta decay if $B(Z,A) - B(Z+1, A) > \Delta = M_n - (M_p + m_e)$. In this case, the energy which would potentially be released in beta decay, i.e. the mass deficit Δ, is insufficient to make up the difference in binding energies. This is the case for many nuclei in our universe because the mass deficit is quite small, i.e. $\Delta = 0.782$ MeV. Hence, the reason why neutrons, which are unstable when free, become stabilised within a nucleus, results from the binding energy spectrum of the nuclei. The neutron is in a potential well too deep for it to climb out of by decaying.

Conversely, we have also seen in Chapter 9 that if beta decay is energetically possible then it does generally occur. Hence we can induce beta instability of a nucleus by increasing the mass of the neutron sufficiently, whilst holding $M_p + m_e$ fixed, so that $B(Z,A) - B(Z+1, A) < \Delta = M_n - (M_p + m_e)$. We are assuming that the strengths of the nuclear and electric forces are also held constant, so that the LHS of this inequality is fixed. The following Table shows the value of $B(Z,A) - B(Z+1, A)$ for the most abundant isotopes of each element up to calcium. This binding energy difference is expressed as a factor by which the mass deficit $\Delta = M_n - (M_p + m_e) = 0.782$ MeV would have to increase to bring about beta instability. The final column shows the corresponding factor by which the neutron mass would have to increase, assuming $M_p + m_e = 938.783$ MeV to be fixed.

Decay	B(Z,A) – B(Z+1, A)	Factor on Δ	Factor on M_n
D → 2p	2.224	2.8	1.0015
He4 → Li4 → p + He3	20.578	26.3	1.0211
Li7 → Be7	1.645	2.1	1.0009
B11 → C11	2.765	3.5	1.0021
C12 → N12	18.121	23.2	1.0185
N14 → O14	5.926	7.6	1.0055

[1] Even gravitational clumping of the neutrons to form large agglomerations of mass would be prevented by the absence of a cooling mechanism.

O16 → F16 → p + O15	15.663	20.0	1.0158
F19 → Ne19	4.020	5.1	1.0034
Ne20 → Na20	14.669	18.8	1.0148
Na23 → Mg23	4.839	6.2	1.0043
Mg24 → Al24	14.661	18.7	1.0148
Al27 → Si27	5.595	7.2	1.0051
Si28 → P28	15.115	19.3	1.0153
P31 → S31	6.179	7.9	1.0057
S32 → Cl32	13.468	17.2	1.0135
Cl25 → Ar35	6.748	8.6	1.0063
Ar40 → K40 → Ca40	1.758	2.2	1.0010
K39 → Ca39	7.313	9.3	1.0070
Ca40 → Sc40	15.102	19.3	1.0152

Thus, as we imagine the neutron mass being slowly increased, the first element to become unstable is lithium, when the neutron is a mere 0.09% more massive – though this is an increase in the mass deficit of 110%. This is shortly followed by argon. However, the instability of lithium-7 and argon-40 is not obviously fatal to the reaction pathways for nucleosynthesis in stars, nor the emergence of life.

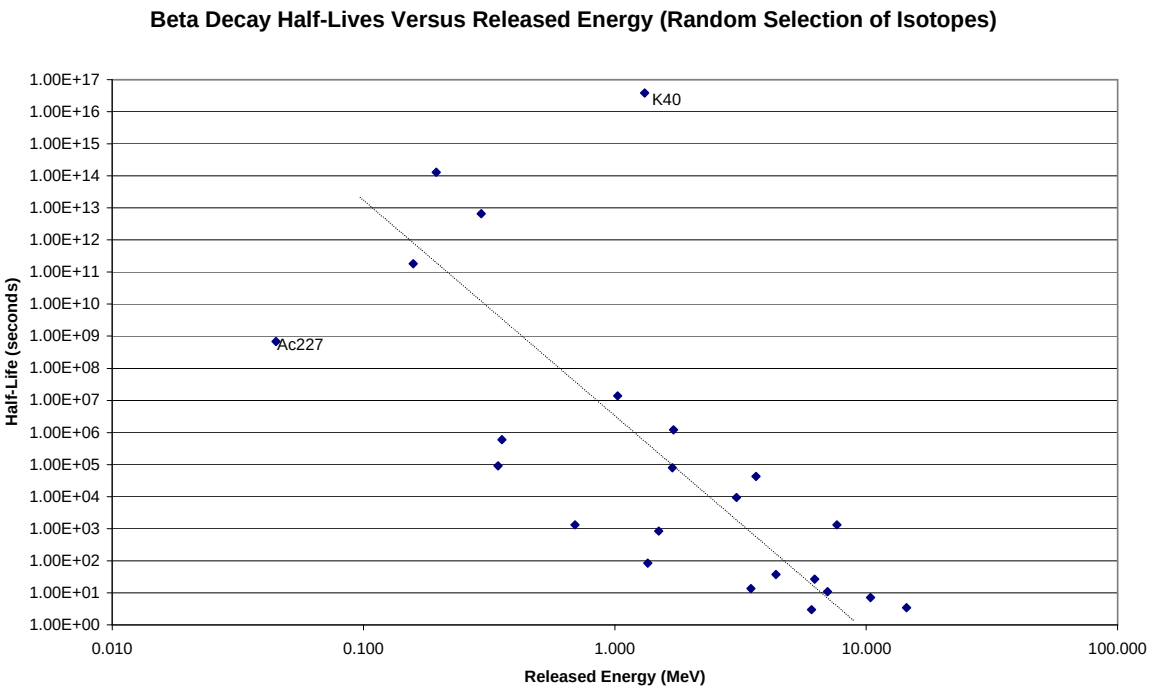
The next isotope to become unstable is the deuteron, at a neutron mass increase of 0.15%, which is a mass deficit increase of 180%. The key elements for life, carbon and oxygen, require an order of magnitude greater increase to become unstable, whereas nitrogen is intermediate. Helium, the stability of which is essential for the formation of heavier elements in stars, is the least sensitive element to neutron mass changes.

The instability of the deuteron is a potential mechanism for preventing all heavier element formation. The deuteron is essential to the reaction pathway to helium, and hence to the heavier elements, both in Big Bang nucleosynthesis and in stars. However, the subsequent reactions, e.g. $p + D \rightarrow {}^{3}_{2}He$, are very rapid and require the deuteron to be stable typically only for a few seconds. Consequently, beta instability of the deuteron can be tolerated without prejudicing the formation of the chemical elements required for life, so long as the deuteron half-life exceeds a few seconds. To explore the likelihood of the beta decay half life being less than a few seconds we have listed a random selection of beta decay half-lives in this universe, below:-

Isotope	**$M_n - (M_p + m_e)$ minus** $B(Z, A) - B(Z+1, A)$ **(MeV)**	**Beta Decay Half-Life (secs)**
${}^{14}_{6}C$	0.157	1.8E11
${}^{16}_{7}N$	10.420	7.1
${}^{19}_{8}O$	6.239	27
${}^{20}_{8}O$	3.472	13.5
${}^{21}_{8}O$	14.464	3.4
${}^{20}_{9}F$	7.024	11

$^{23}_{10}$Ne	4.375	37
$^{32}_{15}$P	1.711	1.2E6
$^{45}_{20}$Ca	1.025	1.4E7
$^{40}_{19}$K	1.311	3.8E16
$^{42}_{19}$K	3.646	4.3E4
$^{43}_{19}$K	1.694	7.9E4
$^{44}_{19}$K	7.633	1300
$^{56}_{25}$Mn	3.050	9400
$^{57}_{25}$Mn	1.352	85
$^{58}_{25}$Mn	6.054	3
$^{98}_{43}$Tc	0.194	1.3E14
$^{99}_{43}$Tc	0.293	6.6E12
$^{222}_{87}$Fr	1.492	840
$^{223}_{87}$Fr	0.693	1320
$^{227}_{89}$Ac	0.045	6.9E8
$^{231}_{90}$Th	0.343	9E4
$^{237}_{92}$U	0.355	5.9E5

These are plotted below:-



As expected, there is a rough correlation between the energy released, i.e. the difference between the mass deficit and the change in binding energy, and the (reciprocal of the) half-life. It is noteworthy that none of the random selection of isotopes examined had a half-life of less than 3 seconds. From the above plot, for a

beta decay half-life to be less than this would most probably require an energy release in excess of ~10 MeV.

Thus, whilst a value of $M_n - (M_p + m_e)$ greater than 2.224 MeV would make beta decay of the deuteron energetically possible, values between 2.224 MeV and ~12 MeV would probably result in deuteron half lives greater than a few seconds. In this case, deuteron instability would not be fatal to the hydrogen burning nuclear reaction sequence in stars. This permits a 15-fold increase in the mass deficit, or ~1.2% increase in the mass of the neutron. Helium, carbon and oxygen would still be stable with such an increase in the neutron mass, though other elements might not be (see the above Table). However, it would appear that a 15-fold increase in the mass deficit, or ~1.2% increase in the mass of the neutron, is closer to the true bound for successful elemental synthesis than that at which the deuteron first becomes unstable [i.e. at a mass deficit increase of x1.8, at a neutron mass increase of 0.15%].

A dilemma faces us with this result. Does the 15-fold tolerance to an increase in mass deficit indicate that there is barely even a Type D coincidence at work? Or, in contrast, does the very slight, 1.2%, increase in neutron mass required to prevent elemental synthesis imply that there is a strong Type B coincidence at work? In as far as the neutron and proton masses are independent, the latter would appear to be a defensible interpretation. This partly translates into the assumed independence of the up and down quark masses. However, since almost all the nucleon mass is generated by the gluon field, it also implies that independent changes to the neutron and proton masses can be achieved by varying the parameters of the appropriate quantum-chromodynamics. Thus, we appear to have a candidate Type B coincidence. Moreover, it is double-sided because $M_p + m_e$ is the lower bound for the neutron mass, as explained in Section 2 above. Thus, our Type B coincidence is,

$$\frac{M_p + m_e}{M_n\,(\text{actual})} = 0.999167 < \frac{M_n}{M_n\,(\text{actual})} < 1.012 \qquad (3.1)$$

where the lower bound originates from stability against electron capture, and the upper bound from stability against beta decay (or having a sufficiently long half-life).

(3.1) is slightly misleading in that it assumes $M_p + m_e$ to be constant. The coincidence is more accurately written as a constraint on the mass deficit,

$$0 < \frac{\Delta = M_n - \left(M_p + m_e\right)}{\Delta_{\text{actual}}} < 15 \qquad (3.2)$$

This is a more accurate way of writing the constraint since it permits all three masses M_n, M_p, m_e to be changed in general. Whilst (3.2) makes the coincidence seem far less impressive, this is essentially because it fails to give credit for Δ_{actual} being so remarkably small compared with the nucleon mass in this universe.

4. Stability of the Electron Structure in Atoms

When the structure of atoms first became apparent to physicists, early in the twentieth century, the great conundrum was how the orbiting electrons avoided spiralling into the nucleus by the emission of electromagnetic radiation. This problem was solved by

quantum mechanics. The recognition that there is a lowest possible energy state for a bound electron, and its concrete formulation via the Schrodinger equation, eliminated the problem. The Schrodinger equation defines what energy levels are available for electrons to occupy. However, the Schrodinger equation alone does not ensure that atoms will have their familiar chemistry. This requires atoms whose electrons form the hierarchical shell structure which gives rise to the regularities of the Periodic Table.

The chemical properties of the elements result from the electronic shell structure. The Pauli exclusion principle is essential to enforce this hierarchical shell structure. The exclusion principle forbids two electrons from occupying the same quantum state. This principle prevents all the electrons from piling together into the lowest energy state. It is the combination of the Schrodinger equation and the exclusion principle which gives rise to atoms exhibiting a rich variety of chemical properties – including, crucially, the ability to bond and form molecules. Chemistry, and hence biology, are possible only because of the nature of quantum mechanics, as exemplified in this case by the Schrodinger equation and the exclusion principle.

Essential though the Schrodinger equation and the exclusion principle are to the stability and binding properties of atoms, they are not usually considered to be anthropically conditioned. The Schrodinger equation is the manifestation for this particular problem of the general precepts of quantum mechanics. Similarly, the exclusion principle is a consequence of quantum field theory and the requirement for Lorentz covariance, i.e. special relativity. Thus they both derive from basic principles of theory rather than from the values adopted by particular universal constants. Of course one may wonder whether basic physical theory is itself anthropically conditioned. Maybe so, but this would take us into waters deeper than we intend to venture.

The existence of a lowest electron energy state, together with the exclusion principle, is essential for the stability and chemical properties of atoms - but they are not sufficient. We will say more about the stability conditions shortly. Firstly, we note that an atom consisting of electrons orbiting around a central nucleus implies that the electrons must be light compared with the nucleus. But why should a central nucleus with orbiting electrons be so important? Could we envisage matter to be made of atoms whose electrons and ‘nuclei’ were of comparable mass, hence giving rise to atoms of a more uniform distribution? This question is addressed in the following sub-section.

4.1 Solidity

In as far as matter is to be equated with mass, atoms consist almost entirely of empty space. Virtually all their mass is concentrated into a tiny fraction, $\sim 10^{-15}$, of their volume – the nucleus. It terms of their mass density, the electrons form a tenuous cloud around the nucleus. It is the extent of this electron cloud which defines the size of the atom. But if the electron cloud is so tenuous, why cannot two atoms overlap and merge? Or, at least, why does not solid matter exhibit a spongy compressibility corresponding to this apparently soft boundary of the atom? The answer, of course, is that the electron clouds of neighbouring atoms repel each other due to the electrostatic force between like charges. The fact that the electrostatic force is strong at short range

means that atoms approximate[2] to billiard balls in their reluctance to overlap. This is the origin of the rigidity of solid matter. It comes about because the positive charge is strongly localised at the centre of the atom. This leaves the electrons alone to inhabit the outer regions of all atoms, and hence provide a mutually repulsive barrier. The reason why the positive charge is so clearly separated from the negatively charged electrons is that the proton mass is so much greater than the electron mass. Hence, it appears that the phenomenon of solidity results from the fact that $M_p/m_e >> 1$.

A counter-example will emphasise the importance of the relative masses of the nucleus and the electron. Suppose that the electron and the proton had the same mass. Consider the hydrogen atom. Rather than the electron orbiting a relatively massive proton, both the electron and the proton would now orbit around their common centre of mass – which would be midway between them. The Schrödinger wavefunctions of the electron and the proton would be identical. Consequently, the expectation value of the charge density at any point would be zero. In this situation there is no longer any electrostatic repulsion operating between two neighbouring hydrogen atoms. Two hydrogen atoms would be able to overlap or pass through each other unimpeded. Of course, the exclusion principle provides a limit to this. If the two protons are of opposite spin, and the two electrons are also of opposite spin, then the two hydrogen atoms could pass, ghost-like, through each other. If either pair of spin states were the same, the exclusion principle would forbid the atoms to also occupy the same space and the two hydrogen atoms would effectively repel each other. However, if one of the atoms were excited to, say, an n = 2 state, then the two atoms could again share a common centre of gravity, though their spatial wavefunctions would now differ. Note that the two hydrogen atoms in this example would not be bound together, and hence the pair of overlapping atoms is quite distinct from a hydrogen molecule.

In this example we have ignored the effect of the strong nuclear force acting between the two protons. Despite the fact that the strong force is not strong enough the bind two protons, it might be that it would be strong enough to produce a bound state from two overlapping hydrogen atoms analogous to a hydrogen molecule, but about a single positive charge centre. Speculation regarding this is not apposite to our purpose.

However, there is a flaw in the argument that equality of the electron and nucleon masses would prevent solidity in all the elements. This is because, for Z > 1, atoms have nuclei heavier than an electron simply because they contain a number, A, of nucleons. Consequently, a degree of localisation of the nucleus would arise in the heavier elements. How can this be quantified? We need to consider what ‘reduced mass’ enters the Schrödinger equation for an N-body problem. It is easily shown that the total kinetic energy of N particles can be written,

$$\text{K.E.} = \frac{1}{2}M\left|\overline{V}\right|^2 + \sum_{i=1}^{N}\frac{1}{2}m_i\left|\overline{v}'\right|_i^2 \qquad (4.1.1)$$

where M is the total of the N particle masses, m_i, and V is the velocity of their centre of mass. The velocities $\overline{v_i'} = \overline{v}_i - \overline{V}$ are with respect to the centre of mass. (4.1.1)

[2] Since billiard balls are made of atoms, the word “approximately” is, perhaps, inappropriate.

follows from the fact that $\sum_{i=1}^{N} m_i \overline{v}_i' = 0$, by definition of the centre of mass.

Consequently, in (4.1.1) only N – 1 of the N velocities $\overline{v}_i'$ are independent. This is important when we convert (4.1.1) into the (kinetic energy part of the) Schrödinger equation for our N-body system. Only the first N – 1 of the coordinates are changed to those with respect to the centre of mass, the N^{th} coordinate is the centre of mass itself. Choosing to eliminate the N^{th} relative velocity, which we interpret as that of the nucleus, the kinetic energy becomes,

$$\text{K.E.} = \frac{1}{2} M \left|\overline{V}\right|^2 + \sum_{i=1}^{N-1} \frac{1}{2} m_i \left(1 + \frac{m_i}{m_N}\right) \left|\overline{v}'\right|_i^2 + \sum_{i \neq j}^{N-1} \frac{m_i m_j}{m_N} v_i' v_j' \qquad (4.1.2)$$

Thus, when we make the quantum replacements $p_i' = m_i v_i' \rightarrow i\hbar\partial_i'$ we find that cross-terms in $\partial_i'\partial_j'$ occur for $N \geq 3$, which is one reason why a many-body system is problematical to analyse. However, for electrons in orthogonal states we may assume that the cross terms average to zero in some sense. The diagonal terms show that each electron has its own reduced mass when paired with the nucleus, i.e.,

$$\sum_{i=1}^{N-1} \frac{1}{2} m_i \left(1 + \frac{m_i}{m_N}\right) \left|\overline{v}'\right|_i^2 = \sum_{i=1}^{N-1} \frac{1}{2m_i} \left(1 + \frac{m_i}{m_N}\right) \left|p_i'\right|^2 = \sum_{i=1}^{N-1} \frac{1}{2\mu_i} \left|p_i'\right|^2 \qquad (4.1.3)$$

where. $\mu_i = \dfrac{m_i m_N}{m_i + m_N}$ is the reduced mass for each electron with respect to the nucleus. Since the position of the nucleus with respect to the c.g. is given by,

$$\overline{r}_N' = -\frac{1}{m_N} \sum_{i=1}^{N-1} m_i \overline{r}_i' \qquad (4.1.4)$$

the expectation value of the square of the radial distance of the nucleus from the c.g. is,

$$\left\langle r_N'^2 \right\rangle = \frac{1}{m_N^2} \int dV_1 dV_2 ... \left\{ \sum_{i,j=1}^{N-1} m_i m_j \overline{r}_i' \cdot \overline{r}_j' \left|\Psi\right|^2 \right\} = \frac{1}{m_N^2} \sum_{i=1}^{N-1} m_i^2 \left\langle r_i^2 \right\rangle \qquad (4.1.5)$$

for orthogonal states. At the crudest level of approximation, if we take all the electrons to have a typical radius of $\left\langle r_i \right\rangle \approx a_0$ then we have,

$$\left\langle r_N' \right\rangle \approx \frac{\left[\sum_{i=1}^{N-1} m_i^2\right]^{1/2}}{m_N} a_0 = \frac{Z^{1/2} m_e}{A M_p} a_0 \qquad (4.1.6)$$

Thus, if the electron and the nucleons had the same mass, the uncertainty in the position of the nucleus would be less than ~10% of the atomic size for Z > 25. The uncertainty in the position of the nucleus would be less than 15% of the atomic size

for Z > 11. The uncertainty in the position of the nucleus would be less than 20% of the atomic size for Z > 6. On this basis it does not seem entirely out of the question that the nucleus could be adequately localised, and hence give rise to solid matter, even if the electron and the nucleons had the same mass. The only limitation would seem to be that the solid elements would be confined to the heavier elements. However, it appears feasible that carbon and above could be solid.

The preceding analysis has considered only the uncertainty in the position of the nucleus of an isolated atom. Of greater interest as regards solidity is the uncertainty in the atom's position with respect to neighbouring atoms. A crude model suffices. Consider an atom in a crystal. If the atom is displaced from its equilibrium position by Δx, its potential energy can be written

$$\text{P.E.} = -\frac{1}{2}k\Delta x^2 \qquad (4.1.7)$$

where k represents the combined 'spring' stiffness due to the atom's interactions with its neighbours. For a simple oscillator, the average kinetic energy of the atom will be comparable with (the magnitude of) its potential energy, so we have,

$$\text{K.E.} = \frac{\Delta p^2}{2M} = \frac{1}{2}k\Delta x^2 \qquad (4.1.8)$$

where M is the mass of the atom. But the uncertainties in the position and momentum are related by,

$$\Delta p.\Delta x \geq \frac{\hbar}{2} \qquad (4.1.9)$$

Substituting (4.1.8) into (4.1.9) gives,

$$\Delta x^2 \geq \frac{\hbar}{2\sqrt{Mk}} \qquad (4.1.10)$$

Thus, the more massive the atom, and/or the more tightly it is bound to its neighbours by a large k, the smaller is its positional uncertainty, Δx. To confirm that this relation conforms to expectations in this universe, we note that a typical binding energy of an atom in a crystal is of order 1 eV. Also, the typical atomic radius is $a_0 \sim 0.5$ Angstrom. Assuming that displacing an atom by one atomic radius effectively frees it from the lattice, substitution of these data into (4.1.7) gives $k \sim 8 \times 10^4$ MeV.fm^{-2}. Using the actual nucleon mass (~938 MeV) in (4.1.10) thus gives an uncertainty in atomic position of about 0.1 Angstrom. This is a reasonably small fraction of the atomic size, and confirms that such an atom can be regarded as localised to within an accuracy of about 20% of the atomic radius.

More generally, we may argue as follows. Crudely approximate the outer electrons as equivalent to those of a hydrogen atom with Z = 1, i.e. assume that the inner electrons screen N – 1 of the N nuclear charges. The atomic size is thus about the Bohr radius,

$$a_0 = \frac{\hbar^2}{\mu e^2} = \frac{\hbar c}{\alpha \mu c^2} \tag{4.1.11}$$

where μ is the reduced mass. The outer electron is thus bound by an energy of,

$$B = \frac{e^2}{2a_0} = \frac{\hbar c}{2} \cdot \frac{\alpha}{a_0} = \frac{1}{2}\alpha^2 \mu c^2 \tag{4.1.12}$$

Since the outer electron is the agent by which the atom is bound to its neighbours, it is reasonable to equate breaking the atomic bond with ionising the outer electron (at least roughly). We again assume that breaking of the atomic bond is effectively accomplished at a displacement of one atomic radius, a_0. Thus we have,

$$\frac{k}{2} a_0^2 \approx \frac{\hbar c}{2} \cdot \frac{\alpha}{a_0} \Rightarrow k a_0^3 \approx \alpha \hbar c \tag{4.1.13}$$

This defines the atomic bond strength 'k'. Substitution into (4.1.10) gives the atomic position uncertainty as,

$$\Delta x \approx \sqrt{\frac{\hbar}{2}} \cdot \frac{a_0^{3/4}}{(M \alpha \hbar c)^{1/4}} \tag{4.1.14}$$

Dividing by a_0 gives the positional uncertainty as a fraction of the atomic size. Substituting for a_0 from (4.1.11) gives,

$$\frac{\Delta x}{a_0} \approx \frac{1}{\sqrt{2}} \left(\frac{\mu}{M} \right)^{1/4} = \frac{1}{\sqrt{2}} \left(\frac{A m_e M_p}{(m_e + A M_p)(Z m_e + A M_p)} \right)^{1/4} \approx \frac{1}{\sqrt{2}} \left(\frac{m_e}{A M_p} \right)^{1/4} \tag{4.1.15}$$

where the last approximation applies only for $m_e << M_p$.

Differentiating the exact form of expression in (4.1.15) shows that the maximum fractional uncertainty occurs for $\frac{m_e}{M_p} = \frac{A}{\sqrt{Z}}$ and takes the value,

$$\left. \frac{\Delta x}{a_0} \right|_{MAX} = \frac{1}{\sqrt{2} \cdot \sqrt{1 + \sqrt{Z}}} \tag{4.1.16}$$

Thus, the largest possible value for the fractional positional uncertainty is ½, and occurs for hydrogen (Z = A =1) when the electron and proton masses are equal. With the actual values of the masses we have a fractional positional uncertainty, as estimated from (4.1.15), of 0.1 - reasonably consistent with our estimate above.

To achieve the maximum uncertain given by (4.1.16) we must have an electron which is heavier than the nucleons by a factor of $A/\sqrt{Z}$. If we limit the electron mass to the nucleon mass, the maximum uncertainty in position becomes,

$$\left.\frac{\Delta x}{a_0}\right|_{MAX}^{m_e=M_p} = \frac{1}{\sqrt{2}}\left[\frac{A}{(A+1)(A+Z)}\right]^{1/4} \approx \frac{0.639}{(2Z+1)^{1/4}} \qquad (4.1.17)$$

where the latter approximation holds only for Z > 1, i.e. not for hydrogen. Using (4.1.17), i.e. assuming the electron and nucleon masses are the same, the fractional uncertainties are found to be,

Z	$\left.\frac{\Delta x}{a_0}\right\|_{MAX}^{m_e=M_p}$
6 (C)	0.34
11 (Na)	0.29
15 (P)	0.27
20 (Ca)	0.25
26 (Fe)	0.24

We have seen above that a fractional uncertainty of ~0.2 is probably sufficient to result in rigid solids (is this true?). Since the positional uncertainties estimated in the above Table are not very much greater than this, it seems quite likely that rigid solids could occur even in a universe in which the electron had the same mass as the nucleon.

We conclude that the oft-claimed requirement that $m_e/M_p << 1$ in order to permit the phenomenon of rigid solids appears to have been over-stated. On the contrary, we find that rigid solids would probably occur even if $m_e = M_p$ because the nucleus is comprised of many nucleons, not just one. However, we have some doubt about this because it is not clear just how small the fractional positional uncertainty has to be to give rise to rigidity. It has been claimed (Kahn, 1972) that the replication fidelity of DNA required for Dawinian evolution to function does need $m_e/M_p << 1$. I must examine this claim more closely.

4.2 Constraint on the Electrostatic Force Strength in Heavy Elements?

Barrow & Tipler, P.297, make the following argument: The innermost electrons in an atom of atomic number Z have binding energy,

$$B = \frac{Z^2}{2}\alpha^2 m_e c^2 \qquad (4.2.1)$$

(assuming now that $m_e << M_p$). The kinetic energy is numerically equal to (4.2.1), and the potential energy is –2 times this (in accord with the Virial Theorem). Hence, Equ.(4.2.1) shows that the typical velocity of the innermost electrons is $Z\alpha c$. Equ.(4.2.1) is, o course, a non-relativistic approximation which will cease to hold as $Z\alpha c$ approaches unity. Barrow & Tipler's argument appears to be that there is an upper limit to the strength, α, of the electrostatic force given by $Z\alpha \sim 2$, since at this

point the innermost electrons' kinetic energy becomes sufficient to create electron-positron pairs.

In practice, as we have seen in Chapter 9, there is a more restrictive upper bound on the electric force strength due to nuclear instability. However, this nuclear limit assumes that the strong nuclear force remains unchanged in strength as the electric force is increased in strength. It would be of importance if Barrow & Tipler's atomic upper bound on α were correct, since it would apply independently of the nuclear force strength.

However, Barrow & Tipler's argument appears spurious. It is quite irrelevant that the electrons possess kinetic energy in excess of $2m_ec^2$. They cannot emit an electron-positron pair. This is because their energy state is the lowest achievable. There is no lower energy state to move to after the emission of the pair. In other words, pair production is impossible for the same reason that the emission of electromagnetic radiation is impossible and the atom is stable. What *is* true is that the increased propensity for *virtual* pair production will alter the nature of the electron orbitals. But this is merely another way of saying that, for $Z\alpha \to 1$, the innermost electrons are relativistic. (NB: Equ.4.2.1 no longer applies in the relativistic regime. The binding energy does not even vary as Z^2, but as some fractional power such as $Z^{7/3}$ or $Z^{4/3}$).

4.3 How Weak Can The Electrostatic Force Be And Still Have Atoms?

Because the electrostatic (Coulomb) force has infinite range, there are always an infinite number of bound states with an energy spectrum proportional to $1/n^2$, whatever the size of α. However, as α is reduced, the binding energy of any given electron orbital reduces as α^2. Thus, whilst stable atoms exist for any size of α, their ionisation potentials and atomic bond strengths reduce rapidly as the Coulomb force is weakened. The implications of this are manifold. Weaker inter-atomic forces will lead to weaker solids, and this has implications for the typical size scale of solid bodies (e.g. planets, asteroids, life-forms, etc – see Chapter 12). It also means that lower temperatures are required to break atomic bonds, or denature complex molecules such as proteins, with implications for biochemistry (see Chapter ??).