International Journal of Pressure Vessels and Piping 129-130 (2015) 32–42

Contents lists available at ScienceDirect

International Journal of Pressure Vessels and Piping

journal homepage: www.elsevier.com/locate/ijpvp

Solution of the ratchet-shakedown Bree problem with an extra orthogonal primary load

R.A.W. Bradford

University of Bristol, Mechanical Engineering, Bristol BS8 1TR, UK

ARTICLE INFO

Article history:
Received 5 December 2014
Received in revised form
27 February 2015
Accepted 2 March 2015
Available online 11 March 2015

ABSTRACT

The complete shakedown and ratcheting solution is derived analytically for a flat plate subject to unequal biaxial primary membrane stresses and a cyclic secondary bending stress in one in-plane direction (x). The Tresca yield condition and elastic-perfectly plastic behaviour are assumed. It is shown that the results can be expressed in the form of a "universal" ratchet diagram applicable for all magnitudes of orthogonal load. For sufficiently large cyclic bending stresses, tensile ratcheting can occur in the x direction if the x direction primary membrane stress exceeds half that in the orthogonal direction. Conversely, for sufficiently large cyclic bending stresses ratcheting in the x direction will be compressive if the x direction primary membrane stress is less than half that in the orthogonal direction. When the x direction primary membrane stress is exactly half that in the orthogonal direction ratcheting cannot occur however large the cyclic secondary bending stress.

1. Introduction

Structural ratcheting is the phenomenon in which cyclic loading results in irrecoverable structural deformation which increases on each cycle. Two or more types of loading are generally required to result in structural ratcheting, at least one of which must be cycling and at least one of which must be primary. The "ratchet boundary" is the boundary of the region in load space above which ratcheting will occur. The "shakedown boundary" is the boundary of the region in load space below which shakedown to elastic cycling will occur. Analytic solutions for the ratchet and shakedown boundaries can be derived only for sufficiently simple problems. The archetype for all such solutions is that of Bree, Ref. [1]. Bree's analytic solution addresses uniaxial loading of a rectangular cross section, the loading consisting of a constant primary membrane stress and a secondary bending load which cycles between zero and some maximum. The problem is rendered analytically tractable by assuming elastic-perfectly plastic behaviour. When normalised by the yield stress, the primary membrane stress is denoted X whilst the normalised secondary elastic outer fibre bending stress range is denoted Y. The ratchet and shakedown boundaries are thus curves on an X,Y plot.

Whilst only sufficiently simple problems can be solved analytically, the analytical treatment of shakedown/ratcheting problems is undergoing something of a revival at present, in acknowledgement of the fact that there are a number of simple problems which are tractable but for which the solutions have not been presented in the literature. For example, variants on the Bree problem have been solved, including the case when the primary membrane load also cycles, either strictly in-phase or strictly out-of-phase with the secondary bending load, Refs. [2–4], and also the Bree problem with different yield stresses at the two ends of the load cycle, Ref. [5]. These analyses used the same approach as Bree's original analysis, Ref. [1]. However, alternative, "non-cycling", methods for analytical ratchet boundary determination are also emerging, e.g., Refs. [6,7].

This paper presents the complete solution of the Bree problem when an extra primary membrane load acts perpendicularly to the main loading. The detailed definition and formulation of the problem is presented in §2. In §3 the solution method is described, and §4 specifies the distinct stress and strain distributions which contribute to the solution. §5 presents the solution itself, in terms of the shakedown and ratchet boundaries, whilst §6 clarifies the distinct ratcheting regions corresponding to tensile or compressive ratcheting. Finally, the most salient features are reprised in the Conclusions, §7.

E-mail address: RickatMerlinHaven@hotmail.com.

http://dx.doi.org/10.1016/j.ijpvp.2015.03.001
0308-0161/© 2015 Elsevier Ltd. All rights reserved.

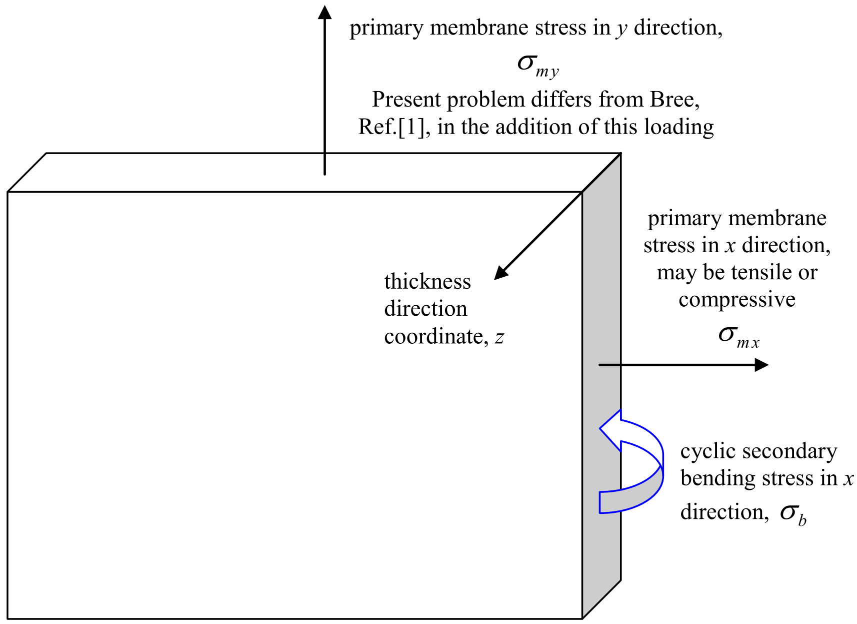

Fig. 1. Geometry and loading.

2. Definition of the problem

The geometry considered is a flat plate in the x,y plane and the problem to be solved is defined by the following loadings (illustrated by Fig. 1).

- A constant primary membrane stress in the x direction (σ_{mx});
- A constant primary membrane stress in the y direction (σ_{my});
- A cycling secondary bending load which would cause an elastic outer fibre bending stress in the x direction of σ_b.

The loading is therefore biaxial. The original Bree problem is regained when the orthogonal stress is zero, $\sigma_{my} = 0$. (In passing we note that Bree's analysis, Ref. [1], is a simplification of the problem which actually motivated it, namely a pressurised thin cylinder with a through-wall thermal gradient. Thus the underlying problem does have primary membrane loads in both directions, i.e., both axial and hoop. However this problem differs from that considered here in that the cylinder is also subject to cycling secondary bending stresses in both directions. Here we impose a cycling secondary bending stress in the x direction only).

The secondary bending stress is envisaged as generated by a uniform through-wall temperature gradient which cycles between zero and some maximum and back again. Bending of the plate about the y axis is restrained so that the temperature gradient causes an x component of bending stress. However, the plate is free

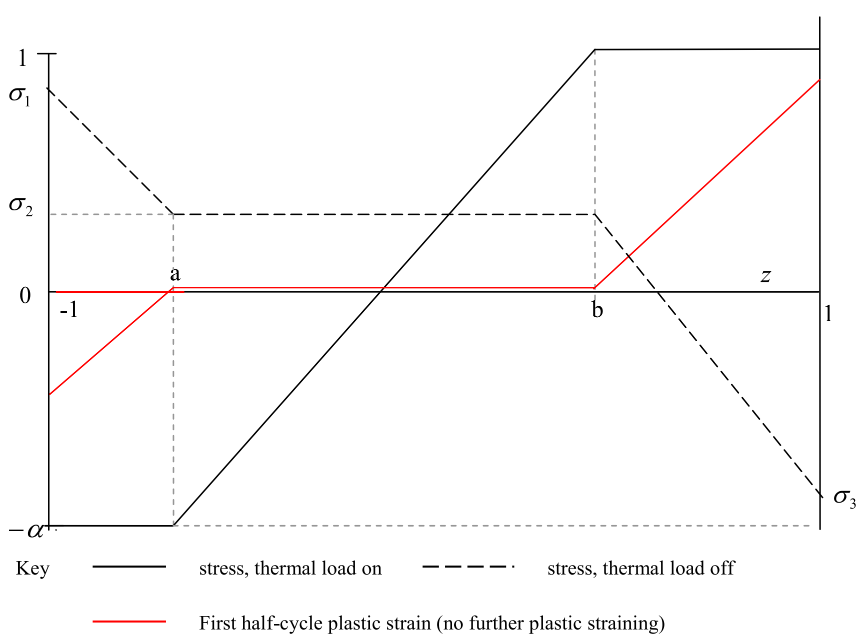

Fig. 2. Axial stress and axial plastic strain versus z (region S2).

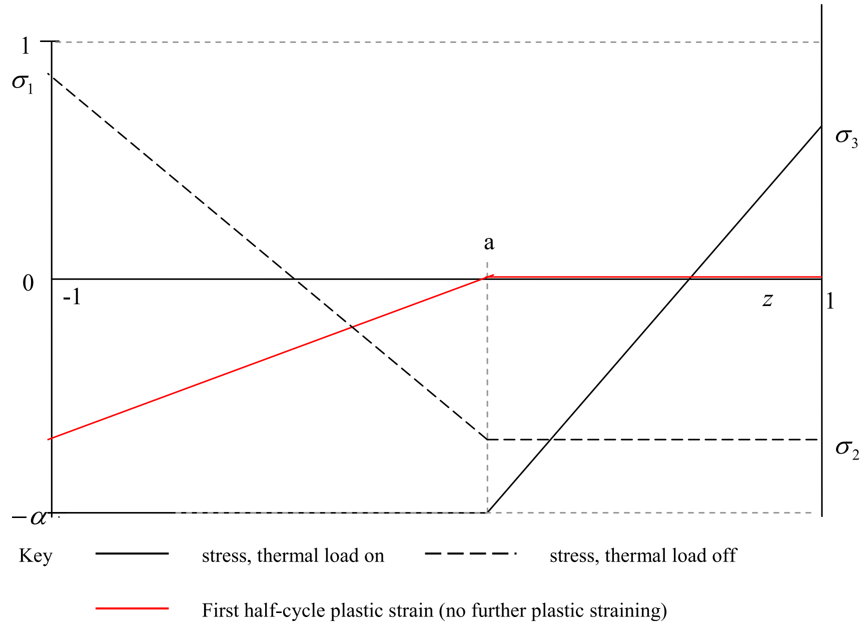

Fig. 3. Axial stress and axial plastic strain versus z (region S3).

to rotate about the x axis and hence there is no y component of bending stress. The plate is also free to undergo overall (membrane) expansion or contraction in all coordinate directions (x,y and the thickness direction, z). Consequently it is possible for ratcheting to occur either as a membrane ratchet strain in the x direction or as a membrane ratchet strain in the y direction (or both).

The x directed membrane stress, σ_{mx}, may be tensile or compressive, but must lie in the range defined by avoidance of plastic collapse. The analysis is presented assuming that the 'orthogonal' membrane stress is tensile, $\sigma_{my} \geq 0$. However this is simply for ease of presentation and the solution for $\sigma_{my} < 0$ will be described in the conclusion.

It will be shown that two distinct types of ratcheting can occur depending upon the relative magnitudes of σ_{mx} and σ_{my}. For large σ_{mx} ratcheting involves tensile ratchet strains in the x direction, but no ratchet strain in the y direction. However, for large σ_{my} and small σ_{mx} ratcheting involves tensile ratchet strains in the y direction and compressive ratchet strains in the x direction.

2.1. Normalised quantities

In terms of the elastic-perfectly plastic yield stress, σ_0, normalised stress axes are defined by,

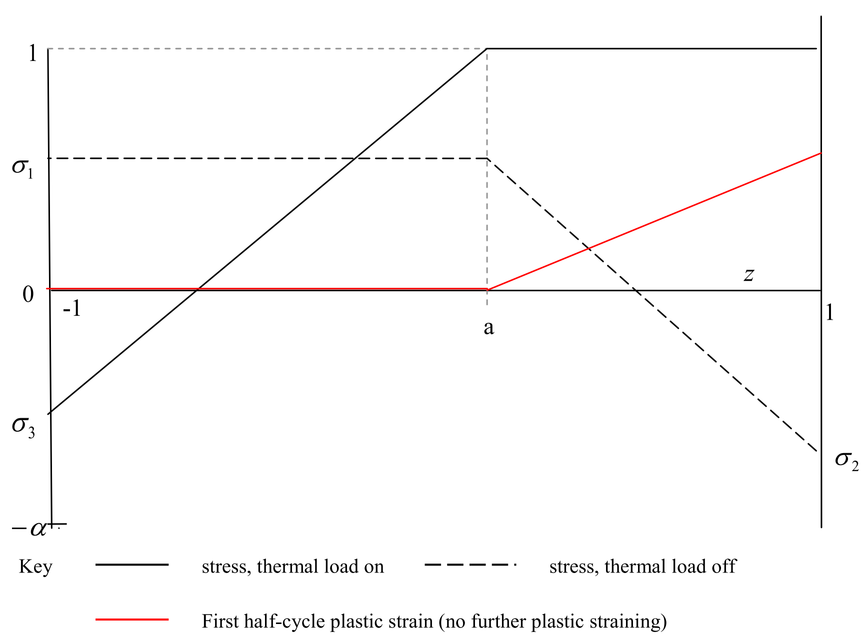

Fig. 4. Axial stress and axial plastic strain versus z (region S1).

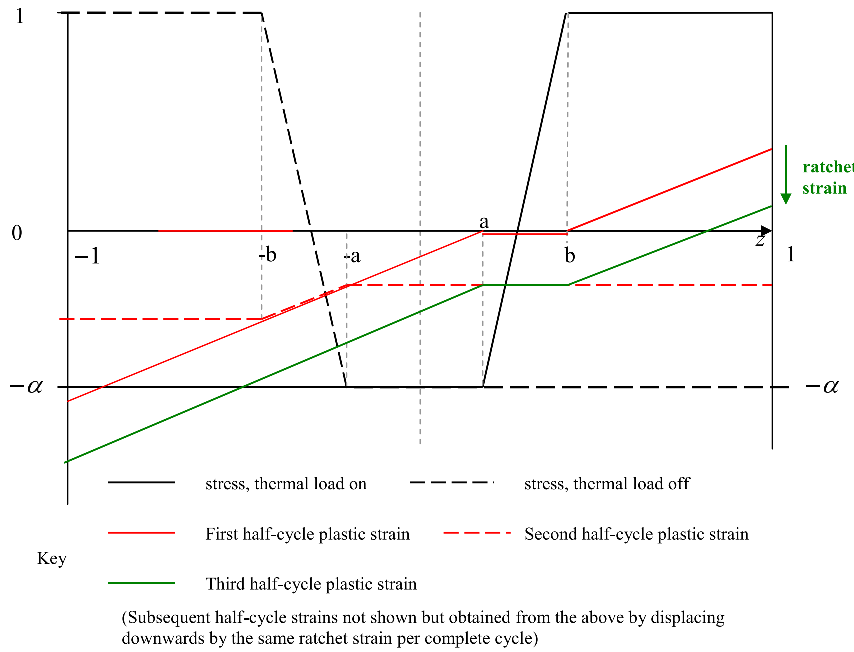

Fig. 5. Axial stress and axial plastic strain versus z (region R4).

$$X = \frac{\sigma_{mx}}{\sigma_0} \quad \text{and} \quad Y = \frac{\sigma_b}{\sigma_0} \tag{1}$$

It is also convenient to define the parameter,

$$\alpha = 1 - \tilde{\sigma}_{my} \quad \text{where,} \quad \tilde{\sigma}_{my} = \frac{\sigma_{my}}{\sigma_0} \tag{2}$$

Hence the solution will be developed in the form of X,Y ratchet diagram plots, which will also depend upon the parameter α.

The x stress at any position through the wall thickness, divided by the yield stress, will be denoted σ. This dimensionless quantity will, in general, vary with the coordinate in the thickness direction, which is denoted z.

This thickness-direction coordinate has its origin at mid-wall and is also made dimensionless by normalising by the half-thickness, so that z varies from -1 on one surface to $+1$ on the other. The thermal bending stress is taken as positive at $z = 1$ (so the thermal strain is negative here).

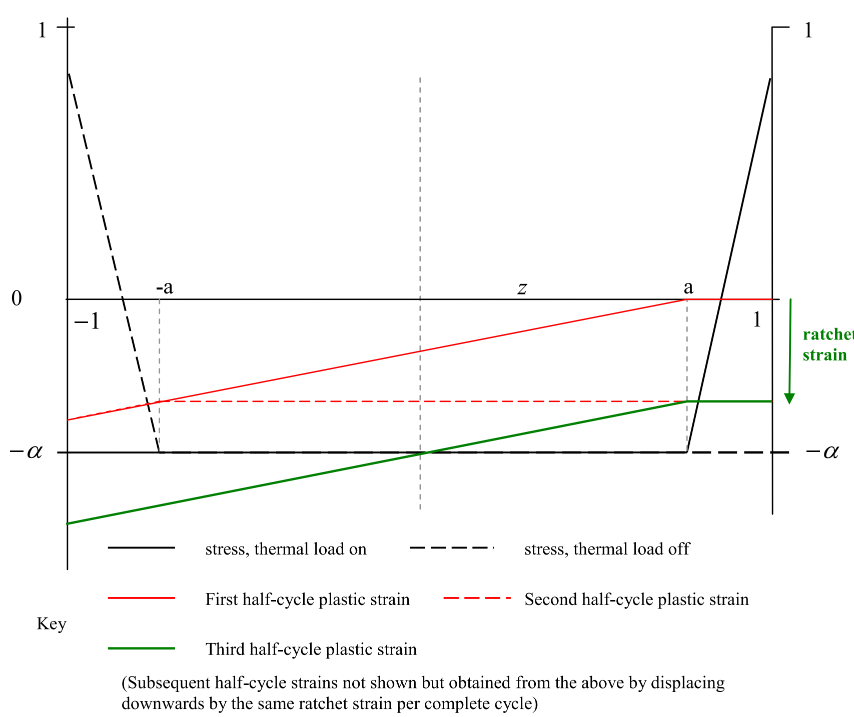

Fig. 6. Axial stress and axial plastic strain versus z (region R3).

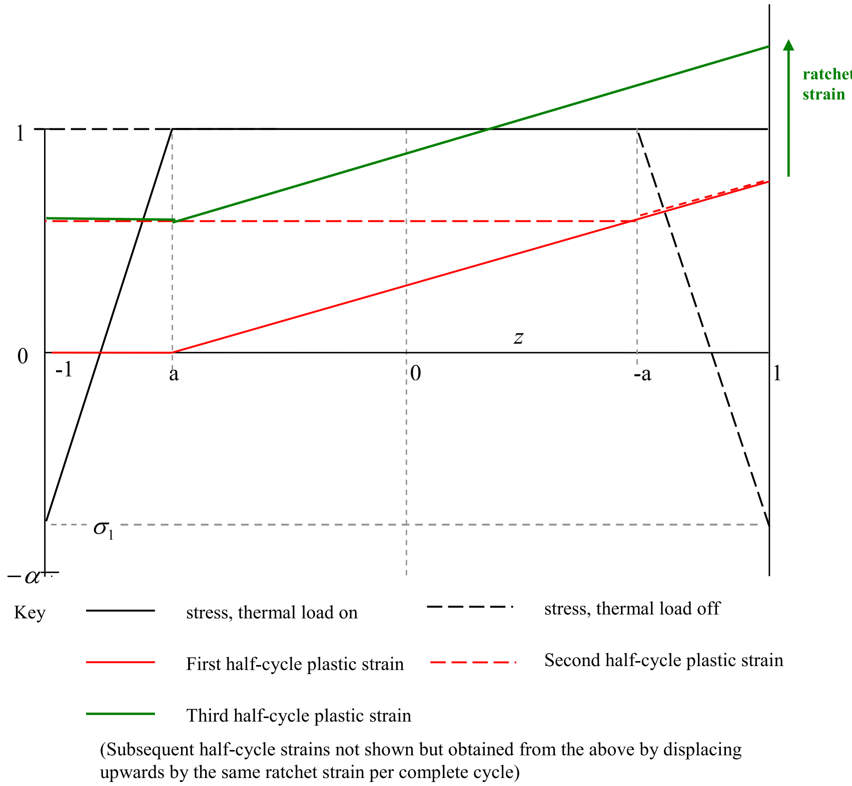

Fig. 7. Axial stress and axial plastic strain versus z (region R1).

All strains will be assumed normalised by the yield strain, $\varepsilon_0 = \sigma_0/E$. Hence, for example, the thermal strain at position z is $-zY$ in our normalisation convention. Henceforth we shall refer to strains normalised in this way simply as “the strain”.

2.2. Formulation of the problem

When the thermal load is acting, the total x strain, ε, is,

$$\text{Thermal ON} \quad \varepsilon = \sigma - \nu\tilde{\sigma}_{my} + \varepsilon_p - zY \tag{3}$$

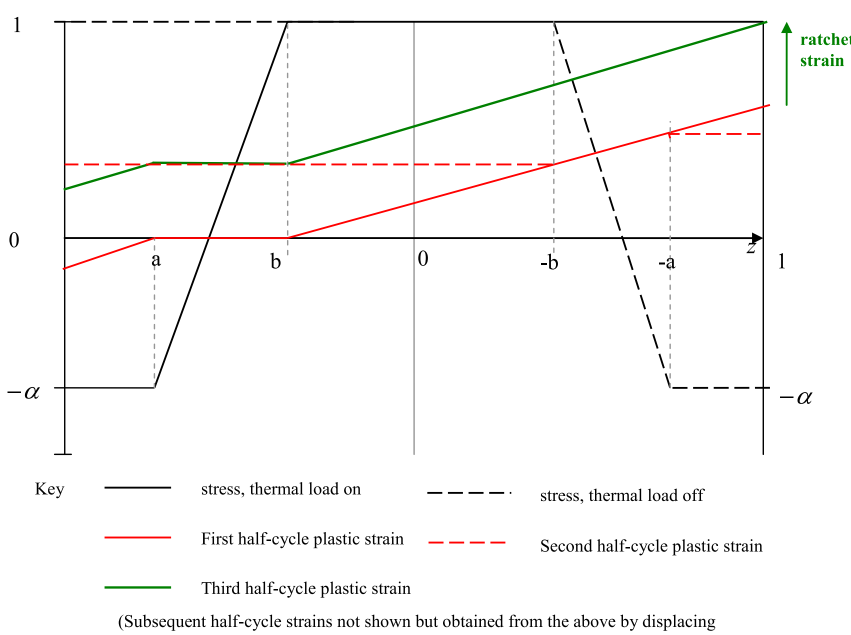

Fig. 8. Axial stress and axial plastic strain versus z (region R2).

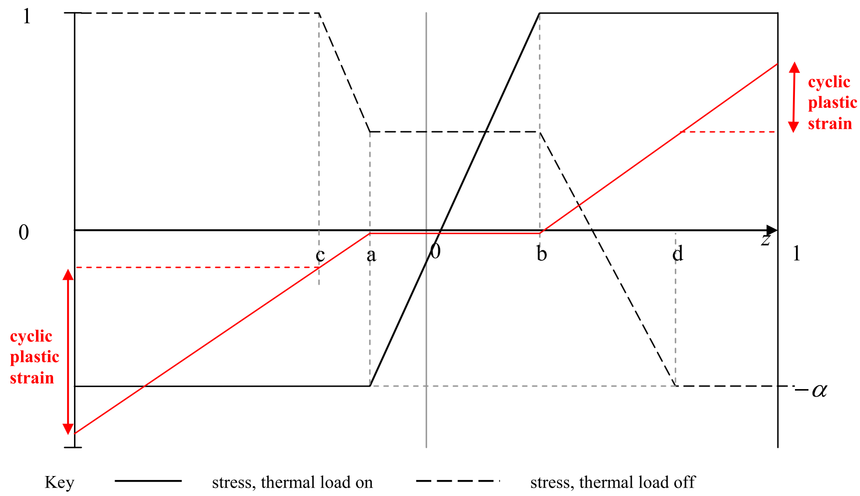

Fig. 9. Axial stress and axial plastic strain versus z (region P).

The sum of the first two terms on the right hand side of (3) is the elastic x strain, whilst ε_p is the plastic x strain and the last term is the thermal strain. When the thermal load is not acting this becomes,

$$\text{Thermal OFF} \quad \varepsilon = \sigma - \nu\tilde{\sigma}_{my} + \varepsilon_p \tag{4}$$

Following §2.1, note that the stresses in (3) and (4) are dimensionless quantities, and the strains are normalised by the yield strain. This normalisation is used throughout and will not be reiterated further.

Equilibrium with the applied x load requires,

$$\text{Equilibrium}: \quad X = \frac{1}{2}\int_{-1}^{+1} \sigma \cdot dz \tag{5}$$

Because the primary loads are taken to be non-cycling, Eq. (5) applies whether the thermal load is acting or not.

The key feature which facilitates the solution is that ε is a membrane strain, i.e., independent of z. Hence, in Eq. (3) the z dependence of $\sigma + \varepsilon_p$ must cancel with that of the thermal strain, $-zY$. In Eq. (4), the z dependence of $\sigma + \varepsilon_p$ must vanish.

Finally, elastic-perfectly plastic behaviour is assumed together with the Tresca yield criterion. It is assumed that the orthogonal stress is tensile and does not result in collapse, $0 \le \tilde{\sigma}_{my} < 1$. Since the z stress is zero, the yield criteria are thus,

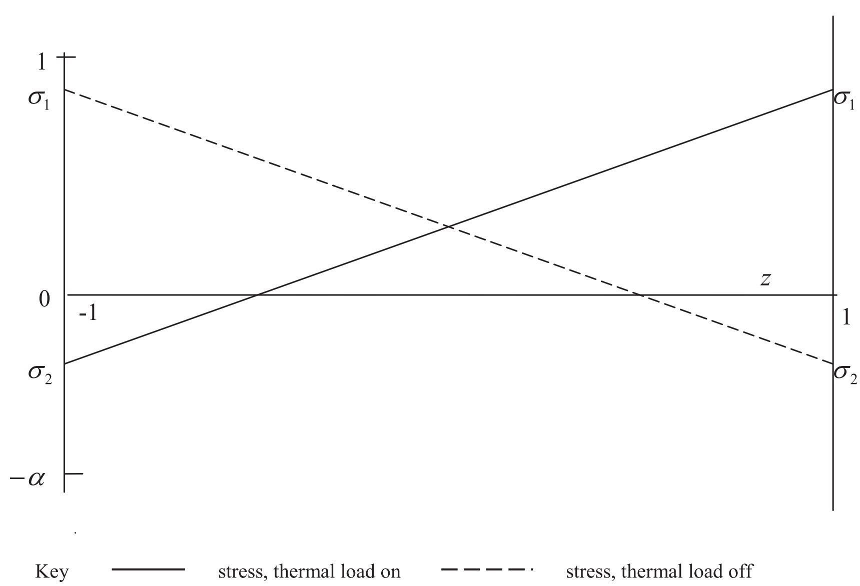

Fig. 10. Axial stress - elastic case (region E).

Table 1
Solution for the stress distributions.

Figure	Parameters defining the stress distributions		
Fig. 2 (S2)	$a = \frac{1-\alpha-2X}{1+\alpha} - \frac{1+\alpha}{2Y}$	$b = \frac{1-\alpha-2X}{1+\alpha} + \frac{1+\alpha}{2Y}$	$\sigma_1 = Y - \alpha$
	$\sigma_2 = \frac{1-\alpha}{2} - Y\left(\frac{1-\alpha-2X}{1+\alpha}\right)$	$\sigma_3 = 1 - Y$	
Fig. 3 (S3)	$a = 1 - 2\sqrt{\frac{X+\alpha}{Y}}$	$\sigma_1 = Y - \alpha$	
	$\sigma_2 = 2\sqrt{Y(X+\alpha)} - Y - \alpha$	$\sigma_3 = 2\sqrt{Y(X+\alpha)} - \alpha$	
Fig. 4 (S1)	$a = 2\sqrt{\frac{1-X}{Y}} - 1$	$\sigma_1 = 1 + Y - 2\sqrt{Y(1-X)}$	
	$\sigma_2 = 1 - Y$	$\sigma_3 = 1 - 2\sqrt{Y(1-X)}$	
Fig. 5 (R4)	$a = \frac{1-\alpha-2X}{1+\alpha} - \frac{1+\alpha}{2Y}$	$b = \frac{1-\alpha-2X}{1+\alpha} + \frac{1+\alpha}{2Y}$	
Fig. 6 (R3)	$a = 1 - 2\sqrt{\frac{X+\alpha}{Y}}$	$\sigma_1 = 2\sqrt{Y(X+\alpha)} - \alpha$	
Fig. 7 (R1)	$a = 2\sqrt{\frac{1-X}{Y}} - 1$	$\sigma_1 = 1 - 2\sqrt{Y(1-X)}$	
Fig. 8 (R2)	$a = \frac{1-\alpha-2X}{1+\alpha} - \frac{1+\alpha}{2Y}$	$b = \frac{1-\alpha-2X}{1+\alpha} + \frac{1+\alpha}{2Y}$	
Fig. 9 (P)	$a = \frac{1-\alpha-2X}{1+\alpha} - \frac{1+\alpha}{2Y}$	$b = \frac{1-\alpha-2X}{1+\alpha} + \frac{1+\alpha}{2Y}$	
	$c = -\frac{1+\alpha}{Y}$	$d = \frac{1+\alpha}{Y}$	
	$\sigma_1 = \frac{1-\alpha}{2} - Y\left(\frac{1-\alpha-2X}{1+\alpha}\right)$		
Fig. 10 (E)	$\sigma_1 = \frac{X+Y}{2}$	$\sigma_2 = \frac{X-Y}{2}$	

$$\text{For } \sigma > 0: \quad \sigma = 1 \tag{6a}$$

$$\text{For } \sigma < 0: \quad \sigma = -\alpha \tag{6b}$$

3. Solution method

The method follows the traditional approach of Refs. [1–5]. It is simplest to implement by translating the above algebraic relations into a geometrical description of the piece-wise linear stress and plastic strain distributions. The key to this is the requirement that ε be independent of z. The rules that permit construction of the distributions are.

1) If the thermal load is acting, then, at all points z,
   - ➢ Either, the slope of the σ versus z graph is zero and the slope of the ε_p versus z graph is Y,
   - ➢ Or, the slope of the σ versus z graph is Y and the slope of the ε_p versus z graph is zero.
2) If the thermal load is not acting, then, at all points z,
   - ➢ Either, the slope of the σ versus z graph is zero and the slope of the ε_p versus z graph is also zero,
   - ➢ Or, the slope of the σ versus z graph is $-Y$ and the slope of the ε_p versus z graph is Y.
3) If the stress is in the elastic range, $-\alpha<\sigma<1$, then the plastic strain, ε_p, is unchanged from its value on the last half-cycle.

Table 2
Solution for the region boundaries.

Boundary between	X,Y Equation of curve	X',Y' Equation of curve
Fig. 3/10	$Y = \alpha + X$	$Y' = X'$
Fig. 4/10	$Y = 1 - X$	$Y' = 1 - X'$
Fig. 2/3	$Y = \frac{(1+\alpha)^2}{4(\alpha+X)}$	$Y' = \frac{1}{4X'}$
Fig. 2/4	$Y = \frac{(1+\alpha)^2}{4(1-X)}$	$Y' = \frac{1}{4(1-X')}$
Fig. 2/9	$Y = 1 + \alpha$	$Y' = 1$
Fig. 3/6	$Y = 4(\alpha + X)$	$Y' = 4X'$
Fig. 5/6	$Y = \frac{(1+\alpha)^2}{4(\alpha+X)}$	$Y' = \frac{1}{4X'}$
Fig. 5/9	$Y = \frac{(1+\alpha)^2}{2(1-\alpha-2X)}$	$Y' = \frac{1}{2(1-2X')}$
Fig. 4/7	$Y = 4(1 - X)$	$Y' = 4(1 - X')$
Fig. 7/8	$Y = \frac{(1+\alpha)^2}{4(1-X)}$	$Y' = \frac{1}{4(1-X')}$
Fig. 8/9	$Y = \frac{(1+\alpha)^2}{2(2X+\alpha-1)}$	$Y' = \frac{1}{2(2X'-1)}$
Point A	$X = \frac{1-3\alpha}{4}; \quad Y = 1 + \alpha$	$X' = \frac{1}{4}; \quad Y' = 1$
Point B	$X = \frac{3-\alpha}{4}; \quad Y = 1 + \alpha$	$X' = \frac{3}{4}; \quad Y' = 1$
Point C	$X = \frac{1-\alpha}{2}; \quad Y = \frac{1+\alpha}{2}$	$X' = \frac{1}{2}; \quad Y' = \frac{1}{2}$

4. The possible stress and strain distributions

There are nine qualitatively distinct stress and plastic strain distributions obeying the rules defined in §3, taking into account both loading conditions with and without the thermal load acting. These nine cases are found simply by exhaustive exploration of what qualitatively different types are possible. The resulting stress and plastic strain distributions, plotted against thickness coordinate, z, are shown schematically in Figs. 2–10. The Figure captions indicate which type of region they correspond to on the X,Y ratchet diagram, where E = elastic, S = shakedown, P = stable cyclic plasticity, and R = ratcheting.

Note that the algebraic expressions for the dimensions a,b,c,… and stresses σ_1,σ_2,… will be different for the different Figures. These expressions are given in Table 1. The manner in which these quantities are found is illustrated in §5, which also derives the shakedown and ratchet boundaries corresponding to each Figure, as appropriate.

5. The shakedown-ratchet diagrams

It is clear from Figs. 2–10 which type of behaviour they represent. What is not immediately obvious is the region to which they correspond on the X,Y ratchet diagram. However these regions can be deduced by repeated application of the rules of §3 together with the equilibrium condition, Eq. (5). The derivations are illustrated in §5.1 and §5.2, below, for a couple of cases, the rest are done in like manner.

The general solution for arbitrary α is given in Table 2 in the form of algebraic expressions for all the boundary curves. Plotting these curves on the X,Y axes leads to the ratchet diagrams, illustrated by Figs. 11 and 12 for α values of 0.1 and 0.5 respectively. These Figures show explicitly which region corresponds to which stress distribution (Figs. 2–10). Note that the range of X is from $-\alpha$ to 1, the former corresponding to plastic collapse under compressive x stress and the latter corresponding to plastic collapse under tensile x stress.

By rescaling the X and Y axes, a single “universal” ratchet diagram, applicable for all values of α, can be obtained. To this end define,

$$X' = \frac{X+\alpha}{1+\alpha} \quad \text{and} \quad Y' = \frac{Y}{1+\alpha} \tag{7}$$

All the boundary curves when plotted as X',Y' are independent of α, as shown explicitly in Table 2. The resulting universal ratchet diagram is shown as Fig. 13. The range of X' is 0–1.

There are two disconnected ratcheting regions, separated by a P-type (stable cyclic plasticity) region. The qualitatively distinct nature of these two ratchet regions is discussed in §6. Both regions become asymptotic to the vertical line $X = (1-\alpha)/2$, or equivalently $X' = 1/2$, when X is half the orthogonal stress. This asymptote is a

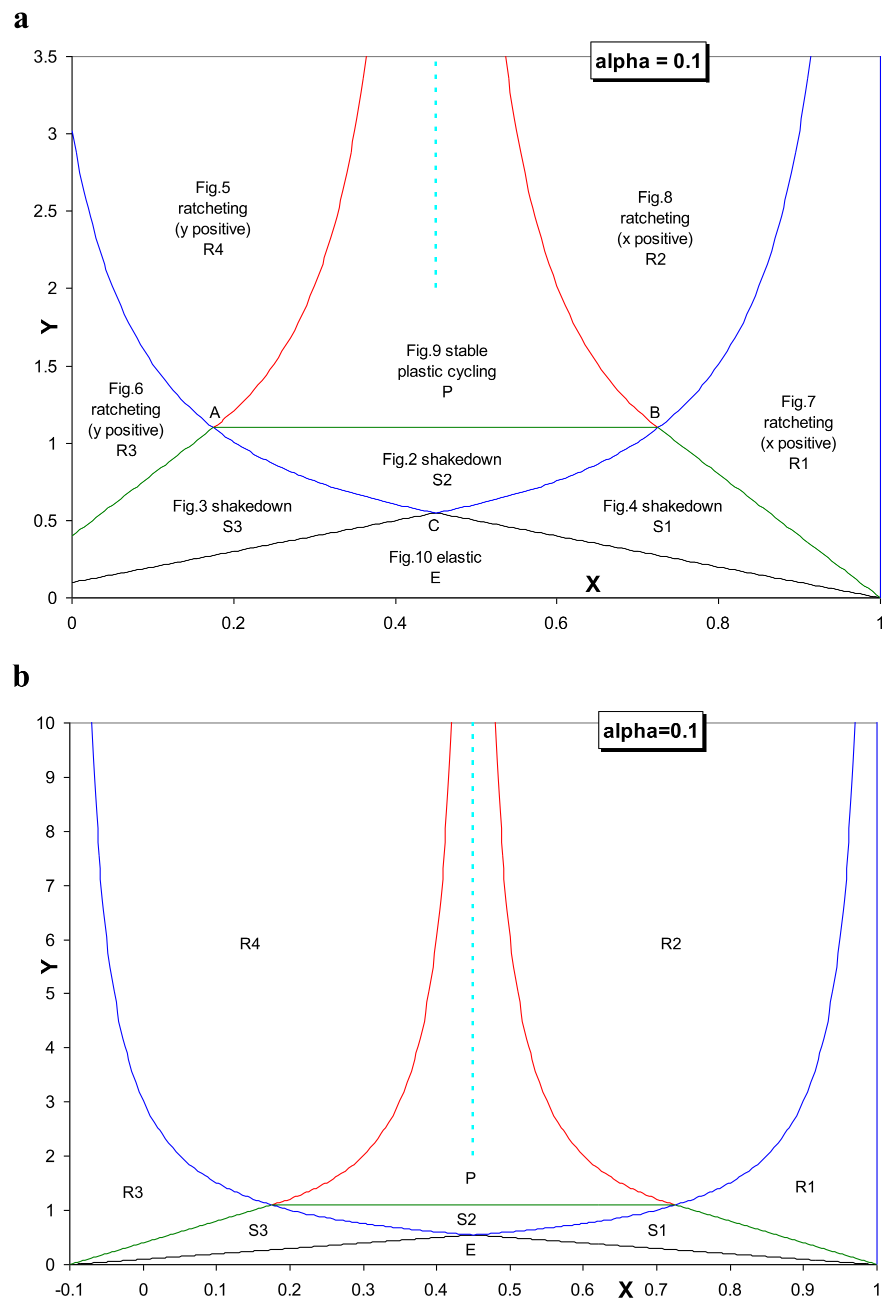

Fig. 11. a)The shakedown–ratchet diagram for $\alpha = 0.1$. b)The shakedown–ratchet diagram for $\alpha = 0.1$ (zoom out).

line of symmetry of the whole shakedown-ratchet diagram. The similarity of the original Bree diagram to Figs. 11–13 to the right of the line of symmetry is clear. The original Bree diagram is regained when $\alpha = 1$, except that we have extended it to compressive x membrane stress.

Note that there is only one S–P boundary, namely the horizontal line at $Y = 1 + \alpha$ (or $Y' = 1$) which extends from point A to point B whose X values are given in Table 2. Near the extremes of X, where collapse is approached in either compression or tension, ratcheting may occur for arbitrarily small Y.

Finally, algebraic expressions for the ratchet strains corresponding to Figs. 5–8 are given in Table 3, as are the cyclic plastic strains corresponding to Fig. 9.

5.1. Illustration–derivation of the boundaries for Fig. 5

As an example of how the algebraic results are derived, consider Fig. 5. The slope of the elastic parts of the stress distributions are $\pm Y$ giving,

$$\frac{1+\alpha}{b-a} = Y \tag{8}$$

Equilibrium, Eq. (5), gives,

$$-\alpha(1+a) + \left(\frac{1-\alpha}{2}\right)(b-a) + 1\cdot(1-b) = 2X \tag{9}$$

Eqs.(8) and (9) can be solved for a,b giving the results shown in Table 1. For Fig. 5 to be valid we require $a > 0$ and $b < 1$. The condition $a > 0$ is required for ratcheting to occur and hence corresponds to the boundary curve with region P. Using the expression for a from Table 1 therefore gives,

$$a = \frac{1-\alpha-2X}{1+\alpha} - \frac{1+\alpha}{2Y} > 0 \tag{10}$$

and this can be re-arranged to give the equation for the boundary between the regions defined by Figs. 5 and 9 as given in Table 2. Similarly the condition $b<1$ provides the boundary with the region defined by Fig. 6 and using the expression for b from Table 1 gives,

$$b = \frac{1-\alpha-2X}{1+\alpha} + \frac{1+\alpha}{2Y} < 1 \tag{11}$$

which can be re-arranged to give the expression for the Fig. 5/6 boundary curve given in Table 2. Finally, since the non-zero slope of the plastic strain curve is Y, Fig. 5 shows that the ratchet strain is given by,

$$\varepsilon_{ratchet} = -2aY \tag{12}$$

Using the expression for a from Table 1 thus produces the expression for the ratchet strain given in Table 3.

5.2. Illustration—derivation of the S–P boundary (Fig. 2)

The a and b parameters of Fig. 2 are given by the same algebraic expressions as for Fig. 5 (the derivation is identical). The three stress parameters$\sigma_1, \sigma_2, \sigma_3$, defining the stress distribution with no thermal load can be found by requiring that the slopes are $-Y$ and using equilibrium, Eq. (5), giving,

$$\frac{\sigma_1-\sigma_2}{1+a} = Y \quad \frac{\sigma_2-\sigma_3}{1-b} = Y \tag{13}$$

$$\left(\frac{\sigma_1+\sigma_2}{2}\right)(1+a) + \sigma_2(b-a) + \left(\frac{\sigma_2+\sigma_3}{2}\right)(1-b) = 2X \tag{14}$$

Solving Eqs. 13 and 14 for σ_1,σ_2,σ_3 provides the expressions given in Table 1. Fig. 2 will be on the boundary of the stable plastic cycling region as $\sigma_1 \to 1$ or as $\sigma_3 \to -\alpha$. As it happens both these conditions are achieved together, the expressions for these stresses given in Table 1 showing that the S–P boundary is simply $Y = 1+\alpha$.

6. The nature of the two ratchet regions

The distinction between the right-hand ratchet regions, R1 and R2, and the left-hand ratchet regions, R3 and R4, lies in which strains are ratcheting, and in which sense, compressive or tensile. This can be elucidated as follows. Consider firstly the right hand regions, R1 and R2. Plastic incompressibility means that,

$$\varepsilon_{yp} + \varepsilon_{zp} + \varepsilon_p = 0 \tag{15}$$

where ε_{yp} and ε_{zp} are the plastic strains in the y and z directions, and ε_p is the plastic strain in the x direction. Taking the difference between (15) over consecutive cycles means that the same relation will hold between their ratchet strains. Consider a region undergoing yielding in x tension, so that the x stress is unity (when normalised by yield). Since the z stress is zero, the (normalised) hydrostatic stress is $(1+\tilde{\sigma}_{my})/3$ and so the (normalised) deviatoric x,y,z stresses are,

$$\hat{\sigma}_x = \frac{2}{3} - \frac{\tilde{\sigma}_{my}}{3}; \quad \hat{\sigma}_y = \frac{2\tilde{\sigma}_{my}}{3} - \frac{1}{3}; \quad \hat{\sigma}_z = -\frac{1}{3} - \frac{\tilde{\sigma}_{my}}{3} \tag{16}$$

In particular this means that $\hat{\sigma}_z \le -\frac{1}{3}$ for $0 \le \tilde{\sigma}_{my} < 1$. Referring to the deviatoric stress plane illustrated by Fig. 14, this means the point in question must lie below the red dashed line (AD). (Recall that the Tresca yield surface crosses the deviatoric stress axes at a value of 2/3). But it must also lie on the Tresca yield surface. Finally it must also lie on a part of the yield surface which has $\hat{\sigma}_x > \hat{\sigma}_y$ since the above equations show this must hold for $\tilde{\sigma}_{my} < 1$. Hence the point in question lies on the line AB, as shown on Fig. 14. Now the normality rule of plastic flow theory states that the plastic strain

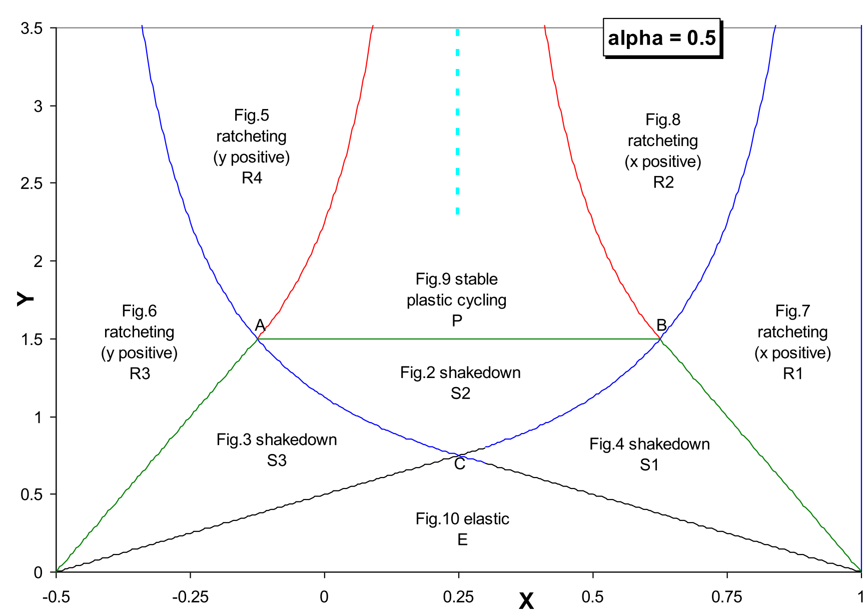

Fig. 12. The shakedown–ratchet diagram for $\alpha = 0.5$.

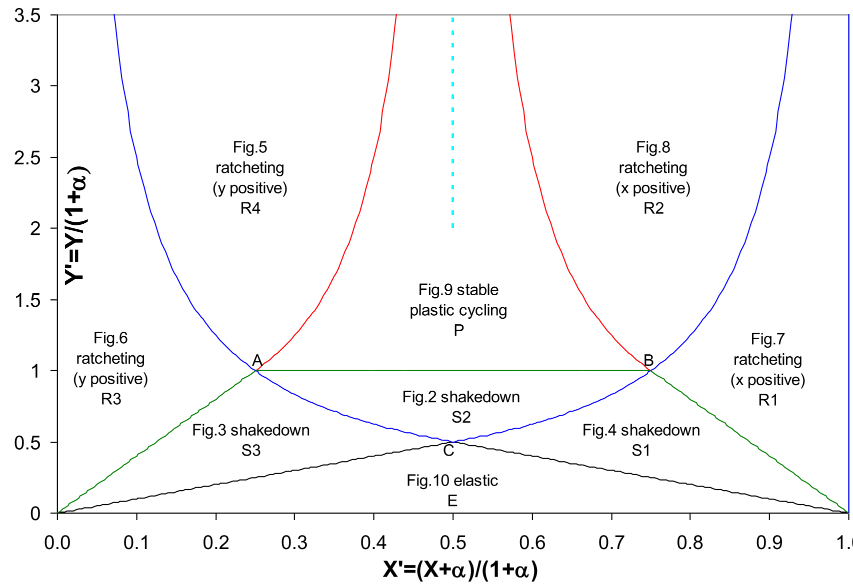

Fig. 13. The 'universal' shakedown–ratchet diagram for arbitraryα.

'vector' in the deviatoric plane must be normal to the yield surface, hence normal to the line AB in Fig. 14. But this means simply that the y plastic strain is zero, and incompressibility is satisfied by the z strain being equal and opposite to the x strain, i.e., $\varepsilon_{zp}+\varepsilon_p = 0$. In other words, regions R1 and R2 correspond to tensile ratcheting in the x direction balanced by thinning of the plate thickness, but no plastic y strains.

Now consider a region undergoing yielding in x compression, so that the x stress is $-\alpha$ (when normalised by the yield stress). Since the z stress is zero, the hydrostatic stress is $(-\alpha + \tilde{\sigma}_{my})/3 = (2\tilde{\sigma}_{my} - 1)/3$ and so the deviatoric x,y,z stresses are,

$$\widehat{\sigma}_x = -\frac{2}{3} + \frac{\tilde{\sigma}_{my}}{3}; \quad \widehat{\sigma}_y = \frac{\tilde{\sigma}_{my}}{3} + \frac{1}{3}; \quad \widehat{\sigma}_z = \frac{1}{3} - \frac{2\tilde{\sigma}_{my}}{3} \tag{17}$$

In particular this means that $\frac{1}{3} \le \widehat{\sigma}_y \le \frac{2}{3}$ for $0 \le \tilde{\sigma}_{my} < 1$. Referring to the deviatoric stress plane of Fig. 14, this means that the point in question must lie to the left of the green dashed line (BC). But it must also lie on the Tresca yield surface. Finally, the above expressions show that it must also lie on a part of the yield surface which has $\widehat{\sigma}_y - \widehat{\sigma}_x$=1. Hence the point in question lies on the line CD, as shown on Fig. 14. Now the normality rule of plastic flow theory states that the plastic strain 'vector' in the deviatoric plane must be normal to the yield surface, hence normal to the line CD in Fig. 14. But this means simply that the z plastic strain is zero, and incompressibility is satisfied by the y strain being equal and opposite to the x strain, i.e., $\varepsilon_{yp}+\varepsilon_p = 0$. But it is clear from the direction of the strain increment, normal to CD on Fig. 14, that it is the y strain which is the tensile ratchet strain for the R3 and R4 regions, whilst the ratchet strains in the x direction are compressive (consistent with Figs. 5 and 6). In these regions the thickness of the plate is unaffected by the ratcheting.

Table 3
Solution for the cyclic plastic or ratchet x strains. For regions R1 and R2, the z ratchet strains are equal and opposite to the x ratchet strains and the y ratchet strains are zero. For regions R3 and R4, the y ratchet strains are equal and opposite to the x ratchet strains and the z ratchet strains are zero.

Figure	Parameters defining the stress distributions
Fig. 5 (R4)	$\varepsilon_{ratchet} = 2Y\left(\frac{2X+\alpha-1}{1+\alpha}\right) + 1 + \alpha$
Fig. 6 (R3)	$\varepsilon_{ratchet} = -2(Y - 2\sqrt{Y(X+\alpha)})$
Fig. 7 (R1)	$\varepsilon_{ratchet} = 2(Y - 2\sqrt{Y(1-X)})$
Fig. 8 (R2)	$\varepsilon_{ratchet} = 2Y\left(\frac{2X+\alpha-1}{1+\alpha}\right) - 1 - \alpha$
Fig. 9 (P)	$\varepsilon_{cyclic} = Y - 1 - \alpha$ (both surfaces)

7. Conclusions

The geometrical similarity between the original Bree diagram and the universal ratchet diagram of Fig. 13 for the present problem may beguile the reader into thinking that the addition of the orthogonal y load has no effect. But this is not the case. Consider, for example, the case of negligible primary x load (i.e., $X = 0$). With no y load the original Bree diagram indicates that stable plastic cycling will occur if $Y > 2$ but that ratcheting will not occur even for arbitrarily large Y. In contrast, suppose that the orthogonal y membrane stress is half yield. Fig. 12 shows that cyclic plasticity will occur for $Y > 3/2$ if $X = 0$ whilst for $Y > 2.25$ there will be ratcheting, a very different result. The additional loading thus has a non-trivial (and deleterious) effect. Moreover, this case with σ_{my} set to half-yield and $\sigma_{mx} = 0$ differs from the case with σ_{mx} set to half yield and $\sigma_{my} = 0$ which produces shakedown for $Y < 2$ and ratcheting for $Y > 2$.

A curious feature of the ratchet diagram, Fig. 13, is that ratcheting will not occur even for arbitrarily large Y if $X' = 0.5$ (though increasingly large cyclic plastic strains will arise). But $X' = 0.5$ occurs if the x membrane stress is half the y membrane stress, i.e., if $X = (1-\alpha)/2$. This is exactly the relationship between the axial and hoop stresses of a thin pressurised cylinder. In a subsequent paper it will be shown that a similar analysis has significant implications for the ratcheting behaviour of a thin pressurised cylinder subject to a cyclic, secondary global bending load.

Finally it is noted that the solution for $-1 < \tilde{\sigma}_{my} \le 0$ is identical provided that the sign of X is reversed, and the signs of all stresses and strains are reversed.

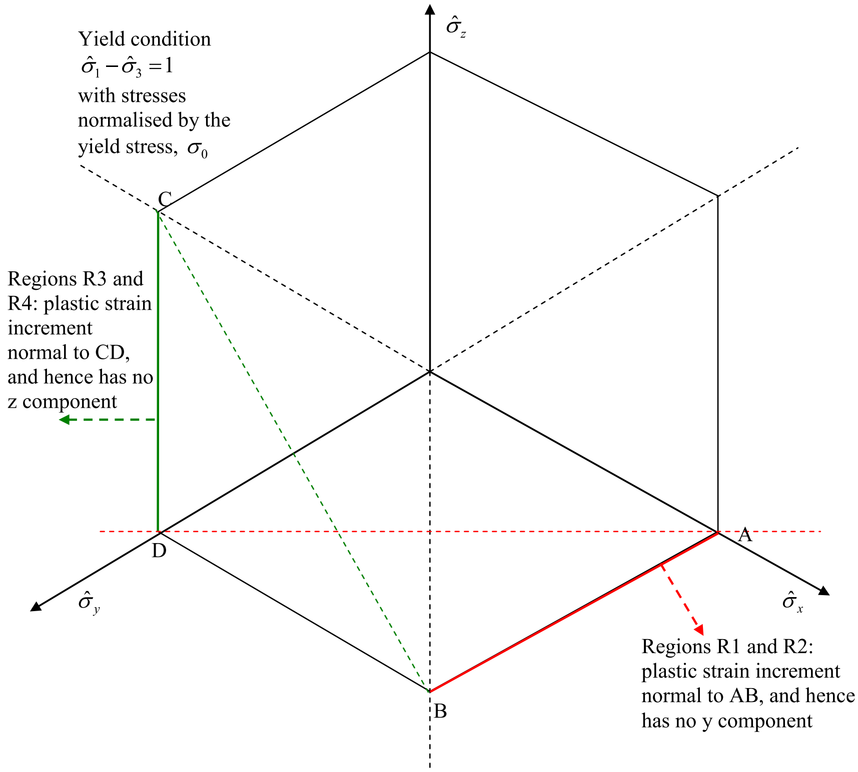

Fig. 14. The deviatoric stress plane and Tresca yield surface.

References

[1] Bree J. Elastic-plastic behaviour of thin tubes subject to internal pressure and intermittent high-heat fluxes with application to fast nuclear reactor fuel elements. J Strain Analysis 1967;2:226–38.

[2] Ng HW, Moreton DN. Ratchetting rates for a Bree cylinder subjected to in-phase and out-of-phase loading. J Strain Analysis Eng Des 1986;21:1–6.

[3] Ng HW, Moreton DN. Alternating plasticity at the surfaces of a Bree cylinder subjected to in-phase and out-of-phase loading. J Strain Analysis Eng Des 1987;22:107–13.

[4] Bradford RAW. The Bree problem with primary load cycling in-phase with the secondary load. Int J Press Vessels Pip 2012;99–100:44–50.

[5] Bradford RAW, Ure J, Chen HF. The Bree problem with different yield stresses on-load and off-load and application to creep ratcheting. Int J Press Vessels Pip 2014;113C:32–9.

[6] Reinhardt W. A non-cyclic method for plastic shakedown analysis. ASME J Press. Vessel Technol 2008;130(3). paper #031209.

[7] Adibi-Asl R, Reinhardt W. Non-cyclic shakedown/ratcheting boundary determination part 1: analytical approach. Int J Pres Ves Pip 2011;88:311–20.