Derivation of the Bree Diagram

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Original Ref: J.Bree, J.Strain Analysis (1967) **2**, 226-238. Below I derive Bree's Figure 3 (reproduced here as Figure 12) in my own manner. It is consierably more detailed than Bree's terse exposition.

Problem: A body is subject to a constant uniaxial primary membrane stress, σ_p, and also a uniaxial secondary bending stress which cycles between zero and σ_t. The secondary bending stress is strain controlled, e.g., being caused by a uniform through-wall temperature gradient. The body is restrained from bending. However, it is possible for a net membrane strain to arise.

Objective: On a diagram of secondary stress range, σ_t, versus primary stress, σ_p, find the region(s) which correspond to ratcheting, and the region which corresponds to cyclic plasticity (creep-fatigue). In the former case determine the ratchet strain per cycle, and in the latter case the cyclic strain range.

Analysis

A through-wall coordinate system, *x*, is defined with $x = 0$ being midwall, and the surfaces being $x = \pm t/2$. The thermal strains are $-\frac{2x}{t}\sigma_t$, where $\sigma_t = E'\alpha\Delta T$. If ε_p is the plastic strain at position *x* and ε is the total strain we therefore have,

(On load) $$E\varepsilon = \sigma + E\varepsilon_p - \frac{2x}{t}\sigma_t \quad (1)$$

The stress, σ, and the plastic strain, ε_p, will generally vary through-thickness, whereas the parameter σ_t is constant. The restraint of bending ensures that the total strain, ε, is also a constant with respect to *x*, though it may vary from one half-cycle to the next. When off-load, the thermal strain is zero and (1) becomes,

(Off load) $$E\varepsilon = \sigma + E\varepsilon_p \quad (2)$$

The yield criterion is simply $|\sigma| = \sigma_y$. If $|\sigma| < \sigma_y$ then the increment of plastic strain at that point is zero. On the first half-cycle, i.e., when load is first applied, any region with $|\sigma| < \sigma_y$ therefore has zero plastic strain. Hence (1) implies that the slope of the stress versus *x* plot must be $\frac{2\sigma_t}{t}$ so that the *x*-dependence of the RHS of (1) cancels.

In the general case, assuming there is some yielding (i.e. that $\sigma_p + \sigma_t > \sigma_y$) the stress after first loading will consist of,

[A] Either, a region in tensile yield, $\sigma = \sigma_y$, plus a region with $|\sigma| < \sigma_y$ and slope $\frac{2\sigma_t}{t}$ (Figure 1),

[B] Or, a region in tensile yield, $\sigma = \sigma_y$, and a separated region in compressive yield, $\sigma = -\sigma_y$, plus a region with $|\sigma| < \sigma_y$ connecting them with slope $\frac{2\sigma_t}{t}$ (Figure 8).

Case [A]

The stress after the first half-cycle is assumed to be as shown in Figure 1. The maximum compressive stress must be such that $\sigma_1 < \sigma_y$. But the requirement that the slope is $\frac{2\sigma_t}{t}$ requires,

$$\frac{\sigma_y + \sigma_1}{\frac{t}{2} - a} = \frac{2\sigma_t}{t} \quad \text{and hence,} \quad \sigma_1 = \left(1 - \frac{2a}{t}\right)\sigma_t - \sigma_y \tag{3}$$

Equilibrium with the applied primary load requires,

$$t\sigma_y - \frac{1}{2}\left(\frac{t}{2} - a\right)(\sigma_y + \sigma_1) = t\sigma_p \quad \text{and hence,} \quad \sigma_1 = \frac{4(\sigma_y - \sigma_p)}{1 - \frac{2a}{t}} - \sigma_y \tag{4}$$

Equating (3) and (4) and solving for *a* gives,

$$1 - \frac{2a}{t} = 2\sqrt{\frac{\sigma_y - \sigma_p}{\sigma_t}} \quad \text{and hence} \quad a = \frac{t}{2}\left\{1 - 2\sqrt{\frac{\sigma_y - \sigma_p}{\sigma_t}}\right\} \tag{5}$$

Since $\sigma_1 < \sigma_y$, (3) implies that $\left(1 - \frac{2a}{t}\right)\sigma_t < 2\sigma_y$. Substituting from the first of (5) gives,

$$2\sqrt{\frac{\sigma_y - \sigma_p}{\sigma_t}}\sigma_t < 2\sigma_y \quad \text{and hence} \quad Y(1 - X) < 1 \tag{6}$$

where we have introduced the normalised stresses,

$$X = \frac{\sigma_p}{\sigma_y} \qquad \text{and} \qquad Y = \frac{\sigma_t}{\sigma_y} \tag{7}$$

(6) establishes that Case [A] lies ***below*** the parabola $Y(1 - X) < 1$ on the Bree diagram (Figure 12). However, we have yet to distinguish between regions S_1 and R_1.

To find the strain distribution after the first half-cycle firstly note that at $x = -a$ we have $\varepsilon_p = 0$ and hence (1) gives,

$$E\varepsilon^1 = \sigma_y + \frac{2a}{t}\sigma_t \tag{8}$$

The superscript on the total strain refers to the first half-cycle. But also, for $x > -a$, (1) gives,

$$E\varepsilon^1 = \sigma_y + E\varepsilon_p^1 - \frac{2x}{t}\sigma_t \tag{9}$$

Substituting (8) in (9) gives $\quad E\varepsilon_p^1 = 2\sigma_t\left(\frac{a}{t} + \frac{x}{t}\right) \quad (10)$

This results in the plastic strain distribution after the first half-cycle, ε_p^1, as given by the black line in Figure 2.

2nd Half-Cycle

The thermal strains have now been removed and so (2) applies. The slope of the stress distribution must therefore be equal opposite to that of $E\varepsilon_p$ at all points. Note that this is the *total* plastic strain accumulated to-date, i.e., $\varepsilon_p^1 + \varepsilon_p^2$. However, any region with non-zero stress slope must be non-yielding in the 2nd half-cycle and hence have $\varepsilon_p^2 = 0$, and hence have opposite slope to the black line in Figure 2. This establishes that the non-yielding region in the 2nd half-cycle has slope $-\frac{2\sigma_t}{t}$. Equilibrium is then sufficient to establish that the stress distribution must be as shown in Figure 3.

To find the plastic strains after the 2nd half-cycle, firstly consider the region $x < a$ where $\sigma = \sigma_y$ and for which (2) gives,

$$E\varepsilon^2 = \sigma_y + E\varepsilon_p^{1+2} \tag{11}$$

where $\varepsilon_p^{1+2} = \varepsilon_p^1 + \varepsilon_p^2$ is the total plastic strain due to the 1st plus 2nd half-cycles. Now consider the region $x > a$ for which (2) gives,

$$E\varepsilon^2 = \sigma + E\varepsilon_p^1 \tag{12}$$

In (12) we have $\varepsilon_p^{1+2} = \varepsilon_p^1$ because $x > a$ is not yielding in the 2nd half-cycle, and note that we know ε_p^1 from (10). Similarly we know the stress from Figure 3, which algebraically is,

$$(x > a) \qquad \sigma = \sigma_y + 2\sigma_t\left(\frac{a}{t} - \frac{x}{t}\right) \tag{13}$$

Substitution of (10) and (13) in (12) gives,

$$E\varepsilon^2 = \sigma_y + \frac{4a}{t}\sigma_t \tag{14}$$

Substitution of (14) in (11) therefore gives,

$$(x < a) \qquad E\varepsilon_p^{1+2} = \frac{4a}{t}\sigma_t \tag{15}$$

For $x > a$ we have $\varepsilon_p^{1+2} = \varepsilon_p^1$ as given by (10). Hence (10) and (15) together give the plot of ε_p^{1+2} shown in Figure 4.

3rd Half-Cycle

The stress distribution now returns to that of Figure 1. The plastic strains in the region $x < -a$ are therefore unchanged from those of Figure 4, since this region is non-yielding on the loading half-cycle. Hence (1) gives,

$$(x < -a) \qquad E\varepsilon^3 = \sigma(x) + E\varepsilon_p^{1+2} - \frac{2x}{t}\sigma_t \tag{16}$$

where $\sigma(x) = \sigma_y + 2\sigma_t\left(\frac{x}{t} + \frac{a}{t}\right)$ from Figure 1, and $E\varepsilon_p^{1+2}$ is given by (15). Substituting in (16) the *x*-dependence cancels, as it must, leaving,

$$E\varepsilon^3 = \sigma_y + \frac{6a}{t}\sigma_t \tag{17}$$

In region $x > -a$ (1) becomes,

$(x > -a)$
$$E\varepsilon^3 = \sigma_y + E\varepsilon_p^{1+2+3} - \frac{2x}{t}\sigma_t \tag{18}$$

Substitution of (17) in (18) gives,

$(x > -a)$
$$E\varepsilon_p^{1+2+3} = 2\sigma_t\left(\frac{x}{t} + \frac{3a}{t}\right) \tag{19}$$

Hence (15) and (19) together give the plastic strain distribution after the 3rd half-cycle, ε_p^{1+2+3}, shown as the red line in Figure 2.

> The effect of the complete cycle comprising the 2nd and 3rd half-cycles is to shift the plastic strain curve upwards uniformly by an amount $\frac{4a}{t}\cdot\frac{\sigma_t}{E}$ (the amount by which the red line in Figure 2 is above the black line).

Repeating the analysis reveals that each unloading half-cycle increases the plastic strain by an amount ε_p^{2n} as shown in Figure 5 (obtained by subtracting the black line in Figure 2 from Figure 4).

Similarly, each loading half-cycle increases the plastic strain by an amount ε_p^{2n+1} as shown in Figure 6 (obtained by subtracting Figure 4 from the red line in Figure 2).

Hence the ratchet strain over a whole cycle is the sum of Figures 5 and 6, which is just the uniform (membrane) strain $\frac{4a}{t}\cdot\frac{\sigma_t}{E}$. This is the ratchet strain.

$$\text{Ratchet Strain} = \frac{4a}{t}\cdot\frac{\sigma_t}{E} \text{ per cycle} \tag{20}$$

<u>Ratcheting Boundary</u>

(6) has established that Case [A] lies below the parabola $Y(1-X)<1$ on the Bree diagram (Figure 12). But we still need to determine the boundary within this region which demarks ratcheting from shakedown. This will be found to be the condition,

Ratcheting:
$$a > 0 \tag{21}$$

This is close to being obvious given that the ratchet strain is $\frac{4a}{t}\cdot\frac{\sigma_t}{E}$, since $a > 0$ is the condition that the ratchet strain be positive and this is almost obvious given that the primary stress is tensile. However, to show this rigorously we consider what would happen if we took $a < 0$.

The equivalent to Figures 1, 2 and 3 can be derived exactly as before and are shown as Figures 7a, b and c. No problem is apparent at this stage. However, see what happens when we attempt to derive the strain after the 2nd half-cycle. Consider the region $-|a| < x < |a|$. Equ.(2) gives,

$$E\varepsilon^2 = \sigma + E\varepsilon_p^1 \tag{22}$$

and the stress from Figure 7c is $\sigma = \sigma_y + 2\sigma_t\left(\frac{a}{t} - \frac{x}{t}\right)$. But not only is this region not yielding during the 2nd half-cycle, so that $\varepsilon_p^2 = 0$, but from Figure 7b we see that the plastic strain increment in the first half-cycle is also zero, $\varepsilon_p^1 = 0$. Consequently (22) apparently gives $E\varepsilon^2 = \sigma = \sigma_y + 2\sigma_t\left(\frac{a}{t} - \frac{x}{t}\right)$. But this conflicts with the requirement that the total strain be constant over the section due to restrained bending. Thus we see that $a > 0$ is not an allowed condition.

To be more precise, $a > 0$ is not allowed if the stress distribution after the 2nd half-cycle looks qualitatively like Figure 7c, i.e., with a yielded region $x < -|a|$. But what if elastic unloading with no yielding prevailed? The stresses obtained from Figure 7a by elastic unloading would be $\sigma_t - \sigma_1$ on $x = -t/2$ and $\sigma_y - \sigma_t$ on $x = t/2$. For elastic loading to prevail we thus require,

$$\sigma_t - \sigma_1 < \sigma_y \quad \text{and} \quad \sigma_y - \sigma_t > -\sigma_y \tag{23}$$

The second of these is simply $\sigma_t < 2\sigma_y$ whereas the first, upon substitution of (3) reduces to,

$$\text{(Shakedown)} \qquad \frac{2a}{t}\sigma_t < 0 \quad \text{which implies} \quad a < 0 \tag{24}$$

Thus we see that elastic unloading does indeed occur for $a < 0$. This is illustrated by the red lines on Figure 7c. This proves that (21) is indeed the ratcheting condition, as claimed, whereas the opposite, (24), gives elastic unloading, i.e., shakedown.

Finally, we note from (5) that the ratcheting condition can be written,

$$\text{Ratcheting:} \quad 1 - 2\sqrt{\frac{\sigma_y - \sigma_p}{\sigma_t}} > 0 \quad \text{or} \quad \frac{\sigma_y - \sigma_p}{\sigma_t} < \frac{1}{4} \quad \text{or} \quad X + \frac{Y}{4} > 1 \tag{25}$$

Hence, for the region below the parabola $Y(1 - X) < 1$, the condition $X + \frac{Y}{4} = 1$ is the boundary between the shakedown and ratcheting regions, S_1 and R_1 on Figure 12. If $X + \frac{Y}{4} > 1$ then ratcheting occurs with ratchet strain $\frac{4a}{t} \cdot \frac{\sigma_t}{E}$ per cycle, where *a* is positive and given by (5). However, if $X + \frac{Y}{4} < 1$ then the structure shakes down with zero ratchet strain per cycle. The ratchet strain is continuous across the boundary $X + \frac{Y}{4} = 1$, however, because on this boundary $a = 0$.

Case (B)

We have still to investigate the alternative case of a region in tensile yield, $\sigma = \sigma_y$, and a separated region in compressive yield, $\sigma = -\sigma_y$, plus a region with $|\sigma| < \sigma_y$ connecting them with slope $\frac{2\sigma_t}{t}$, as illustrated by Figure 8. Given that we have addressed the region below the parabola $Y(1 - X) < 1$, we expect that this Case (B) will be the region above this parabola. In case (A) we found that ratcheting occurred when the initial stress distribution on-load, Figure 1, had its tensile yield zone extending from the negative x value, $-a$, where $a > 0$, to $x = +t/2$. Consequently we guess that Figure 8, as drawn with its tensile yield zone again extending from the negative x value, $-a$, where $a > 0$, to $x = +t/2$, might again produce ratcheting.

The plastic strain distribution, ε_p^1, after the first loading, i.e., after the first half-cycle, can be derived as follows. Equilibrium gives,

$$\sigma_p t = (t/2 + a)\sigma_y - (t/2 - b)\sigma_y = (a+b)\sigma_y \qquad (26a)$$

Or,

$$a + b = Xt \qquad (26b)$$

In the region $-b < x < -a$ the plastic strain is zero and hence we have,

$$E\varepsilon = \sigma(x) - \frac{2x}{t}\sigma_t = -\sigma_y + 2\sigma_y\left(\frac{x+b}{b-a}\right) - \frac{2x}{t}Y\sigma_y \qquad (27)$$

For this to be independent of x we require,

$$b - a = \frac{t}{Y} \qquad (28)$$

(26b) and (28) thus give,

$$\frac{2b}{t} = X + \frac{1}{Y} \quad \text{and} \quad \frac{2a}{t} = X - \frac{1}{Y} \qquad (29)$$

(27) becomes,

$$E\varepsilon = -\sigma_y + 2\sigma_y\left(\frac{b}{b-a}\right) = -\sigma_y + \sigma_y\left(X + \frac{1}{Y}\right)Y = XY\sigma_y \qquad (30)$$

Now consider $x > -a$, we have,

$$E\varepsilon = XY\sigma_y = E\varepsilon_p^1 + \sigma_y - \frac{2x}{t}Y\sigma_y \qquad (31)$$

Hence,

$$\frac{E}{\sigma_y}\varepsilon_p^1 = XY - 1 + \frac{2x}{t}Y \qquad (32)$$

At $x = -a$, using (29), (32) is consistent with $\varepsilon_p^1 = 0$. At $x = t/2$ we get,

At $x = t/2$:

$$\frac{E}{\sigma_y}\varepsilon_p^1 = XY + Y - 1 \qquad (33)$$

Now consider $x < -b$, we have,

$$E\varepsilon = XY\sigma_y = E\varepsilon_p^1 - \sigma_y - \frac{2x}{t}Y\sigma_y \qquad (34)$$

Hence,

$$\frac{E}{\sigma_y}\varepsilon_p^1 = XY + 1 + \frac{2x}{t}Y \qquad (35)$$

At $x = -b$, using (29), (35) is consistent with $\varepsilon_p^1 = 0$. At $x = -t/2$ we get, At $x = -t/2$: $$\frac{E}{\sigma_y}\varepsilon_p^1 = XY - Y + 1 \tag{36}$$

On the negative side of the stress distribution we clearly require that this plastic strain is negative, which gives,

$$Y(1-X) > 1 \tag{37}$$

This establishes that case (B) corresponds to the region above the parabola $Y(1-X) < 1$, just as case (A) corresponds to the region below this parabola. [*Incidentally, requiring that (33) be a positive strain requires* $Y(1+X) > 1$, *which is implied by (37) and adds nothing more*].

The results (32) and (35) permit the plastic strain after the first half-cycle, ε_p^1, to be plotted as shown in Figure 8 by the red lines.

2nd Half-Cycle

Firstly consider what would be implied if the stress in the region $-b < x < -a$ after the 2nd half-cycle (unloading) was not constant. The plastic strain for both half cycles would be zero, and hence $\varepsilon_p^{1+2} = 0$. This would require $E\varepsilon = \sigma(x)$, since the thermal strain has been removed. But this is a contradiction since the LHS is independent of *x*, whereas the RHS is not. We conclude that the stress is constant in the region $-b < x < -a$. We consider later what happens if this stress is less than yield magnitude (see "case C"). For now we assume it equals yield.

But the tensile yield stress must extend beyond $x = -a$ in order to produce a positive load. The simplest distribution which is clearly in equilibrium with the applied primary membrane load is a shown in Figure 9. In fact it is clear that this is the only possibility. This follows since the slope of the line is fixed to be equal and opposite to the slope of $E\varepsilon_p^1$ by virtue of their sum, $E\varepsilon$, being independent of *x*. And the sum of the positions shown as *a* and *b* is fixed by equilibrium, i.e., Equ.(26b). This establishes the stress distribution shown in Figure 9. The plastic strain distribution is found as follows.

Consider firstly the region $a < x < b$. We have, since $\varepsilon_p^2 = 0$ in this region,

$$E\varepsilon = E\varepsilon_p^1 + \sigma(x) = \left(XY - 1 + \frac{2x}{t}Y\right)\sigma_y + \sigma_y - 2\sigma_y\left(\frac{x-a}{b-a}\right) \tag{38}$$

Or, $$\frac{E}{\sigma_y}\varepsilon = \left(XY + \frac{2x}{t}Y\right) - 2\left(\frac{x-a}{b-a}\right) \tag{39}$$

This must be x-independent, but this is guaranteed by (28).So (39) becomes,

$$\frac{E}{\sigma_y}\varepsilon = XY - 2\left(\frac{a}{b-a}\right) = XY + \left(X - \frac{1}{Y}\right)Y = 2XY - 1 \tag{40}$$

Now consider the region $x < a$, we have $E\varepsilon = E\varepsilon_p^{1+2} + \sigma_y$, so that,

$x < a$: $$\frac{E}{\sigma_y}\varepsilon_p^{1+2} = 2(XY - 1) \tag{41}$$

Similarly, in the region $x > b$ we have $E\varepsilon = E\varepsilon_p^{1+2} - \sigma_y$, so that,

$$x > b: \qquad \frac{E}{\sigma_y}\varepsilon_p^{1+2} = 2XY \tag{42}$$

At this stage it is not yet clear that the plastic strain (41) is positive, though this will be proved shortly. The plastic strain distribution after the 2nd half-cycle is shown in Figure 9 by the red lines.

3rd Half-Cycle

The stress as shown in Figure 8 is now re-established. The plastic strain after the third half-cycle is found as follows. Consider firstly the region $-b < x < -a$. We have, since $\varepsilon_p^3 = 0$ in this region,

$$E\varepsilon = E\varepsilon_p^{1+2} + \sigma(x) - \frac{2x}{t}Y\sigma_y = E\varepsilon_p^{1+2} - \sigma_y + 2\sigma_y\left(\frac{x+b}{b-a}\right) - \frac{2x}{t}Y\sigma_y \tag{43}$$

Hence,

$$\begin{aligned}\frac{E}{\sigma_y}\varepsilon &= 2(XY-1) - 1 + 2\left(\frac{x+b}{b-a}\right) - \frac{2x}{t}Y = 2(XY-1) - 1 + 2\left(\frac{b}{b-a}\right) \\ &= 2XY - 3 + \left(X + \frac{1}{Y}\right)Y = 3XY - 2\end{aligned} \tag{44}$$

Now consider the region $x < -b$ for which,

$$E\varepsilon = E\varepsilon_p^{1+2+3} - \sigma_y - \frac{2x}{t}Y\sigma_y \tag{45}$$

implies $$\frac{E}{\sigma_y}\varepsilon_p^{1+2+3} = 3XY - 2 + 1 + \frac{2x}{t}Y = 3XY + \frac{2x}{t}Y - 1 \tag{46}$$

At $x = -b$, and using (29), (46) reproduces that $\frac{E}{\sigma_y}\varepsilon_p^{1+2+3} = \frac{E}{\sigma_y}\varepsilon_p^{1+2} = 2(XY-1)$. At $x = -t/2$ (46) gives,

$$x = -t/2: \qquad \frac{E}{\sigma_y}\varepsilon_p^{1+2+3} = 3XY - Y - 1 \tag{47}$$

Now consider the region $x > -a$ for which,

$$E\varepsilon = E\varepsilon_p^{1+2+3} + \sigma_y - \frac{2x}{t}Y\sigma_y \tag{48}$$

implies $$\frac{E}{\sigma_y}\varepsilon_p^{1+2+3} = 3XY - 2 - 1 + \frac{2x}{t}Y = 3(XY-1) + \frac{2x}{t}Y \tag{49}$$

At $x = -a$, and using (29), (49) reproduces that $\frac{E}{\sigma_y}\varepsilon_p^{1+2+3} = \frac{E}{\sigma_y}\varepsilon_p^{1+2} = 2(XY-1)$. At $x = t/2$ (49) gives,

$$x = t/2: \qquad \frac{E}{\sigma_y}\varepsilon_p^{1+2+3} = 3XY + Y - 3 \tag{50}$$

The plastic strain after the 3rd half-cycle is thus as shown in Figure 8 by the upper red line.

Ratchet Strain

The salient fact about the plastic strain distribution after the 3rd half-cycle is that it lies a uniform (membrane) amount above the plastic strain after the first half cycle. This is established by considering the increases in plastic strain between the 1st and 3rd half-cycles at representative points:-

$$x = -t/2: \qquad \frac{E}{\sigma_y}\Delta\varepsilon_p = (3XY - Y - 1) - (XY - Y + 1) = 2(XY - 1) \tag{51a}$$

$$-b < x < -a: \qquad \frac{E}{\sigma_y}\Delta\varepsilon_p = 2(XY - 1) - 0 = 2(XY - 1) \tag{51b}$$

$$x = +t/2: \qquad \frac{E}{\sigma_y}\Delta\varepsilon_p = (3XY + Y - 3) - (XY + Y - 1) = 2(XY - 1) \tag{51c}$$

So the ratchet strain is,

$$\frac{E}{\sigma_y}\Delta\varepsilon_p = 2(XY - 1) \tag{52}$$

It is now clear that, for this ratchet strain to be positive, we require,

$$XY > 1 \tag{53}$$

Hence, (53) together with (37) establishes that the distributions of Figures 8 and 9 correspond to the ratcheting region R_2 of the Bree diagram, Figure 12, i.e., above *both* parabolas.

Note that by virtue of (29), the ratchet condition (53) can equally be written simply,

$$a > 0 \tag{54}$$

which is exactly the same criterion as for case (A), i.e., (21), as we guessed initially. A summary of what we have shown so far is,

The ratcheting condition is $a > 0$ for both cases (A) and (B), so Figures 1 and 8 are correct as drawn. They correspond to the ratcheting regions R_1 and R_2 on the Bree diagram, Figure 12, respectively.

For case (A), Figure 1, if $a < 0$ then we are in the shakedown region S_1 of the Bree diagram, Figure 12.

For case (A), Figure 1, and $a > 0$ the ratchet strain is $\Delta\varepsilon_p = 2\left[Y - 2\sqrt{Y(1-X)}\right]\frac{\sigma_y}{E}$

For case (B), Figure 8, and $a > 0$ the ratchet strain is $\Delta\varepsilon_p = 2(XY - 1)\frac{\sigma_y}{E}$

But we have left some unfinished business. What does Figure 8 imply if we let *a* be negative? This gives us….

Case (C)

The stress distribution after the 1st half-cycle is now as given by Figure 10. Note that the definition of *a* has been reversed in sign so that *a* is still a positive quantity. We shall see that Figure 10 corresponds to the region P of the Bree diagram, Figure 12. It will be shown to have zero ratchet strain but plastic strain increments at the surfaces which are equal and opposite on each alternate half-cycle. This is the region of 'global' shakedown which does not ratchet but which can cause creep-fatigue damage accumulation at the surfaces due to the cyclically reversing plastic strains. This is the region addressed by an R5V2/3 crack initiation assessment with a hysteresis cycle.

Firstly we derive the plastic strains corresponding to Figure 10, and the values of *a* and *b*. Equilibrium gives,

$$X\sigma_y t = \sigma_y(t/2 - a) - \sigma_y(t/2 - b) \qquad \text{hence} \quad Xt = b - a \tag{55}$$

In the region $-b < x < a$ we have, since the plastic strain is zero,

$$E\varepsilon = -\sigma_y + 2\sigma_y\left(\frac{x+b}{a+b}\right) - \frac{2x}{t}Y\sigma_y \tag{56}$$

The *x*-independence of ε therefore implies,

$$a + b = \frac{t}{Y} \tag{57}$$

So (55) & (57) give,
$$\frac{2b}{t} = \frac{1}{Y} + X \quad \text{and} \quad \frac{2a}{t} = \frac{1}{Y} - X \tag{58}$$

These are the same as (29) except that *a* is reversed in sign. (56) becomes,

$$\frac{E}{\sigma_y}\varepsilon = -1 + 2\left(\frac{b}{a+b}\right) = -1 + \left(\frac{1}{Y} + X\right)Y = XY \tag{59}$$

Now consider the region $x > a$. We have,

$$\frac{E}{\sigma_y}\varepsilon = XY = \frac{E}{\sigma_y}\varepsilon_p + 1 - \frac{2x}{t}Y \qquad \text{hence,} \qquad \frac{E}{\sigma_y}\varepsilon_p = XY + \frac{2x}{t}Y - 1 \tag{60}$$

This is consistent with $\varepsilon_p(a) = 0$ whilst at $x = t/2$ we get,

$$\frac{E}{\sigma_y}\varepsilon_p(t/2) = XY + Y - 1 \tag{61}$$

Now consider the region $x < -b$. We have,

$$\frac{E}{\sigma_y}\varepsilon = XY = \frac{E}{\sigma_y}\varepsilon_p - 1 - \frac{2x}{t}Y \qquad \text{hence,} \qquad \frac{E}{\sigma_y}\varepsilon_p = XY + \frac{2x}{t}Y + 1 \tag{62}$$

This is consistent with $\varepsilon_p(-b) = 0$ whilst at $x = -t/2$ we get,

$$\frac{E}{\sigma_y}\varepsilon_p(-t/2) = XY - Y + 1 \tag{63}$$

But this plastic strain must be negative, since it is on the negative side of the stress distribution and there has been no previous plastic straining. So $XY - Y + 1 < 0$, which gives,

$$Y(1 - X) > 1 \tag{64}$$

So, like case (B), case (C) = Figure 10 lies *above* the parabola $Y(1 - X) = 1$.

<u>2nd Half-Cycle</u>

After the thermal load is removed, the stress in the region $-b < x < a$ must be constant (flat). If it were not, then it would be elastic and hence the plastic strain increment on the 2nd half-cycle would be zero. But the plastic strain in the 1st half-cycle is zero also. And the thermal strain has become zero after the 2nd half-cycle. So this would mean that $E\varepsilon = \sigma(x)$, but this is a contradiction since ε is *x*-independent. Hence, the stress must be constant in region $-b < x < a$.

The most general stress distribution after the 2nd half-cycle is therefore as shown in Figure 11. We shall derive the value of the stress, σ_0, in region $-b < x < a$, and also the positions $-c$ and $+c'$ which denote the extent of the regions at yield in Figure 11. It will be found that $c' = c$, but this is not obvious at this stage.

Firstly consider region $-c < x < -b$. The plastic strain increment during unloading is zero, so the total plastic strain equals that for the 1st half-cycle, which is given by (62) in this region. Hence the relation $E\varepsilon = E\varepsilon_p + \sigma$ gives,

$$-c < x < -b: \qquad \sigma = E\varepsilon - E\varepsilon_p = E\varepsilon - \left(XY + \frac{2x}{t}Y + 1 \right)\sigma_y \tag{65}$$

But at $x = -c$ the stress is σ_y so that

$$E\varepsilon = \left(XY - \frac{2c}{t}Y + 2 \right)\sigma_y \tag{66}$$

So (65) becomes,

$$-c < x < -b: \qquad \frac{\sigma}{\sigma_y} = 1 - \frac{2(c + x)}{t}Y \tag{67}$$

Hence,

$$\frac{\sigma_0}{\sigma_y} = 1 - \frac{2(c - b)}{t}Y \tag{68}$$

This specifies the 'core' stress, σ_0, if we can find *c* [and given that we know *b* from (58)]. Now consider region $a < x < c'$. Following the same reasoning we have,

$$a < x < c': \qquad \sigma = E\varepsilon - E\varepsilon_p = E\varepsilon - \left(XY + \frac{2x}{t}Y - 1 \right)\sigma_y \tag{69}$$

But at $x = c'$ the stress is $-\sigma_y$ so that

$$E\varepsilon = \left(XY + \frac{2c'}{t}Y - 2 \right)\sigma_y \tag{70}$$

Equating (66) and (70) gives,

$$\frac{c+c'}{t}=\frac{2}{Y} \tag{71}$$

Using (70), (69) becomes,

$a<x<c'$: $$\frac{\sigma}{\sigma_y}=-1+\frac{2(c'-x)}{t}Y \tag{72}$$

Hence, $$\frac{\sigma_0}{\sigma_y}=-1+\frac{2(c'-a)}{t}Y \tag{73}$$

It is easily shown that (73) follows from (68) given (71) and (58), and so provides no new information. The plastic strains are established as follows,

From (66),

$x<-c$: $$\frac{E}{\sigma_y}\varepsilon_p^{1+2}=\frac{E}{\sigma_y}\varepsilon-1=XY-\frac{2c}{t}Y+1 \tag{74}$$

From (62),

$-c<x<-b$: $$\frac{E}{\sigma_y}\varepsilon_p^{1+2}=\frac{E}{\sigma_y}\varepsilon_p^{1}=XY+\frac{2x}{t}Y+1 \tag{75}$$

Now consider the region $-b<x<a$. Since the stress is less than yield magnitude, and this region was also unyielded in the 1st half-cycle, the plastic strain is zero.

$-b<x<a$: $$\frac{E}{\sigma_y}\varepsilon_p^{1+2}=0 \tag{76}$$

From (60),

$a<x<c'$: $$\frac{E}{\sigma_y}\varepsilon_p^{1+2}=\frac{E}{\sigma_y}\varepsilon_p^{1}=XY+\frac{2x}{t}Y-1 \tag{77}$$

From (70),

$x>c'$: $$\frac{E}{\sigma_y}\varepsilon_p^{1+2}=\frac{E}{\sigma_y}\varepsilon+1=XY+\frac{2c'}{t}Y-1 \tag{78}$$

Note that (74-78) are consistent at the boundaries between the regions, by virtue of (58). Finally we employ equilibrium. Note that the slope of the stress plot is the same in regions $-c<x<-b$ and $a<x<c'$ because of (67) and (69). Hence the sum of the contributions of these regions to the force is zero, and equilibrium can be written,

$$\sigma_y(t/2-c)+\sigma_0(a+b)-\sigma_y(t/2-c')=Xt\sigma_y \tag{79}$$

Substituting the expression (68) for σ_0 and using (57) for $(a+b)$, this becomes,

$$\frac{t}{Y}+c'-3c+2b=Xt \tag{80}$$

But substituting for c' in terms of c from (71), this becomes,

$$\frac{t}{Y}+\left(\frac{2t}{Y}-c\right)-3c+\left(X+\frac{1}{Y}\right)t=Xt \tag{81}$$

Which simplifies to, $c = \frac{t}{Y}$ (82a)

And from (71) we conclude that, $c' = c = \frac{t}{Y}$ (82b)

It follows that the plastic strains at the surfaces, from (74) and (78) become,

$$\frac{E}{\sigma_y}\varepsilon_p^{1+2}(-t/2) = XY - 1 \tag{83}$$

$$\frac{E}{\sigma_y}\varepsilon_p^{1+2}(+t/2) = XY + 1 \tag{84}$$

But we know that the surface strain $\varepsilon_p^{1+2}(-t/2)$ must be negative because the plastic strain in region $-c < x < -b$ slopes *upward* in order to become *zero* for $-b < x < a$. Hence, from (83),

$$\frac{E}{\sigma_y}\varepsilon_p^{1+2}(-t/2) = XY - 1 < 0$$

In other words, $XY < 1$ (85)

This contrasts with case (B) which lay above this hyperbola. Since case (C) lies below this hyperbola, but above $Y(1 - X) = 1$, it lies in the regions P or S_2 on the Bree diagram, Figure 12. It is clear that, for Figure 10 to become Figure 11 on removal of the thermal stress we must also have $\sigma_t > 2\sigma_y$. This establishes that case (C) can be identified with region P alone.

What we have not yet shown is that there is no ratchet strain in this case. For this we need to consider the 3rd half-cycle.

But first note that (82a) and (58) mean that the core stress, (68), becomes simply,

$$\frac{\sigma_0}{\sigma_y} = XY \tag{86}$$

<u>3rd Half-Cycle</u>

The stress now returns to that of Figure 10. The region $-b < x < a$ still has zero plastic strain, so,

$$E\varepsilon = \sigma(x) - \frac{2x}{t}\sigma_y = -\sigma_y + 2\sigma_y\left(\frac{x+b}{a+b}\right) - \frac{2x}{t}\sigma_y \tag{87}$$

The RHS of (87) is independent of *x*, by virtue of (58), as it should be. Hence it gives,

$$\frac{E}{\sigma_y}\varepsilon = XY \tag{88}$$

For $x > a$ we have,

$$\frac{E}{\sigma_y}\varepsilon = \frac{E}{\sigma_y}\varepsilon_p^{1+2+3} + 1 - \frac{2x}{t}Y \tag{89}$$

And so, $$\frac{E}{\sigma_y}\varepsilon_p^{1+2+3} = \frac{E}{\sigma_y}\varepsilon - 1 + \frac{2x}{t}Y = XY + \frac{2x}{t}Y - 1 \tag{90}$$

Hence, $$\frac{E}{\sigma_y}\varepsilon_p^{1+2+3}(t/2) = XY + Y - 1 \tag{91}$$

Note that (90) and (91) are the same as the plastic strains after the 1st half-cycle, (60) and (61), showing that there has been no ratchet strain and that a stable hysteresis cycle has developed.

For $x > -b$ we have,

$$\frac{E}{\sigma_y}\varepsilon = \frac{E}{\sigma_y}\varepsilon_p^{1+2+3} - 1 - \frac{2x}{t}Y \tag{92}$$

And so, $$\frac{E}{\sigma_y}\varepsilon_p^{1+2+3} = \frac{E}{\sigma_y}\varepsilon + 1 + \frac{2x}{t}Y = XY + \frac{2x}{t}Y + 1 \tag{93}$$

Hence, $$\frac{E}{\sigma_y}\varepsilon_p^{1+2+3}(-t/2) = XY - Y + 1 \tag{94}$$

Note that (93) and (94) are the same as the plastic strains after the 1st half-cycle, (62) and (63), showing that there has been no ratchet strain and that a stable hysteresis cycle has developed.

The plastic strains after the 2nd half-cycle in the regions $-c < x < -b$ and $a < x < c$ are the same as (93) and (90) respectively. So the only regions in which the plastic strains change after the first half-cycle are the regions within the distance *c* of either surface. The plastic strain ranges at the surfaces are (1st – 2nd half-cycles):-

At $x = -t/2$: $$\frac{E}{\sigma_y}\Delta\varepsilon_p = 1 + XY - Y - (XY - 1) = 2 - Y \tag{95}$$

At $x = +t/2$: $$\frac{E}{\sigma_y}\Delta\varepsilon_p = -1 + XY + Y - (XY + 1) = Y - 2 \tag{96}$$

The cyclic plastic zone is of depth $c = \frac{t}{Y}$ from each surface. It seems odd that the depth of the plastic zone should *decrease* as the thermal stress is *increased*, but this is the prediction. Of course, this is only true provided we remain within the region $XY < 1$, i.e., within the region P of the Bree diagram, Figure 12. So we have $1/Y > X$ and hence $c > Xt$, which seems more reasonable.

Nevertheless, for a fixed primary stress, the depth of the cyclic plastic zone decreases as the boundary $XY = 1$ is approached from below, reaching a minimum when ratcheting starts. At this point the 'core' stress, σ_0, reaches yield and so the tensile yield zone is greater than t/2 in extent both with and without the thermal stress (Figures 1 and 3).

Case (A): $a > 0$, $\sigma_1 < \sigma_Y$

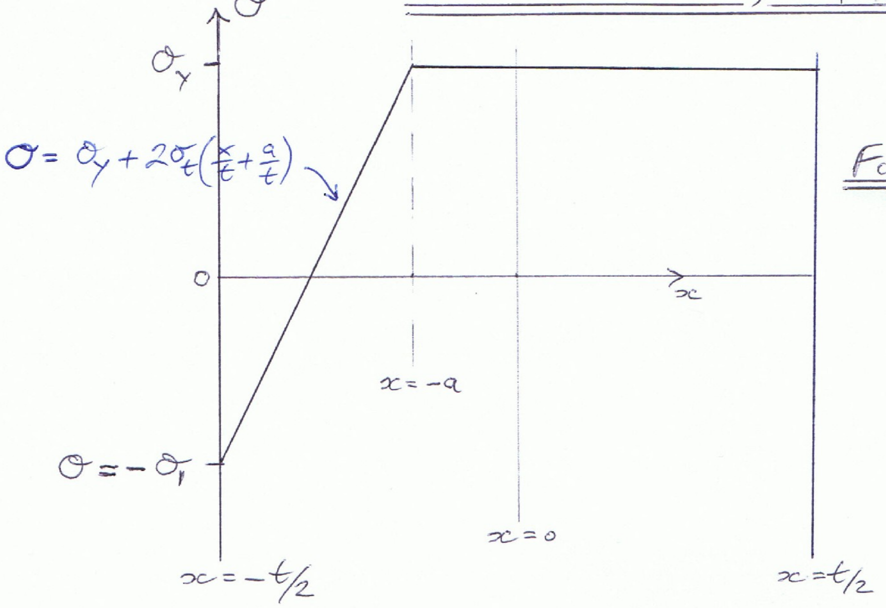


Fig. 1

Stress & Plastic Strain after 1st Half-Cycle & after 3rd Half-Cycle

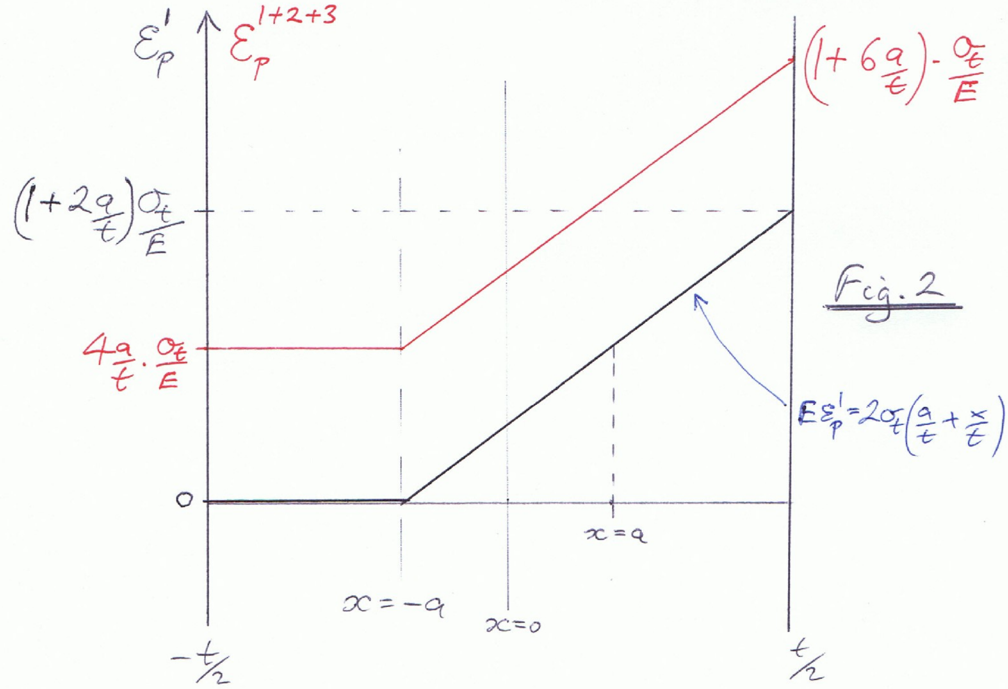
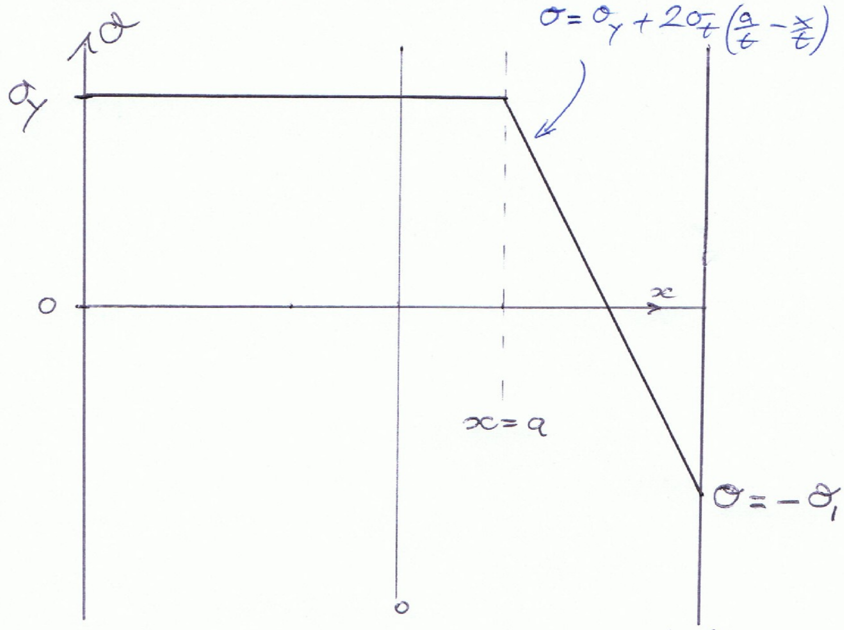


Fig. 2

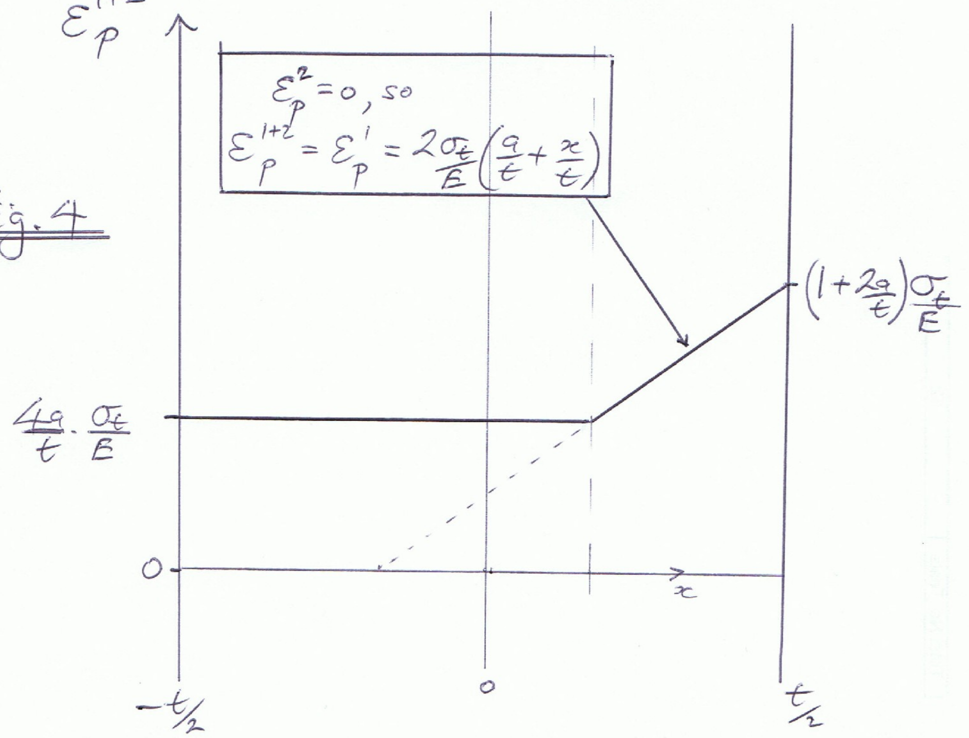
Case (A): $a > 0$, $\sigma_1 < \sigma_Y$

Fig. 3



Stress & Plastic Strain after 2nd Half Cycle

Fig. 4



Case (A): $a > 0$, $\sigma_1 < \sigma_Y$

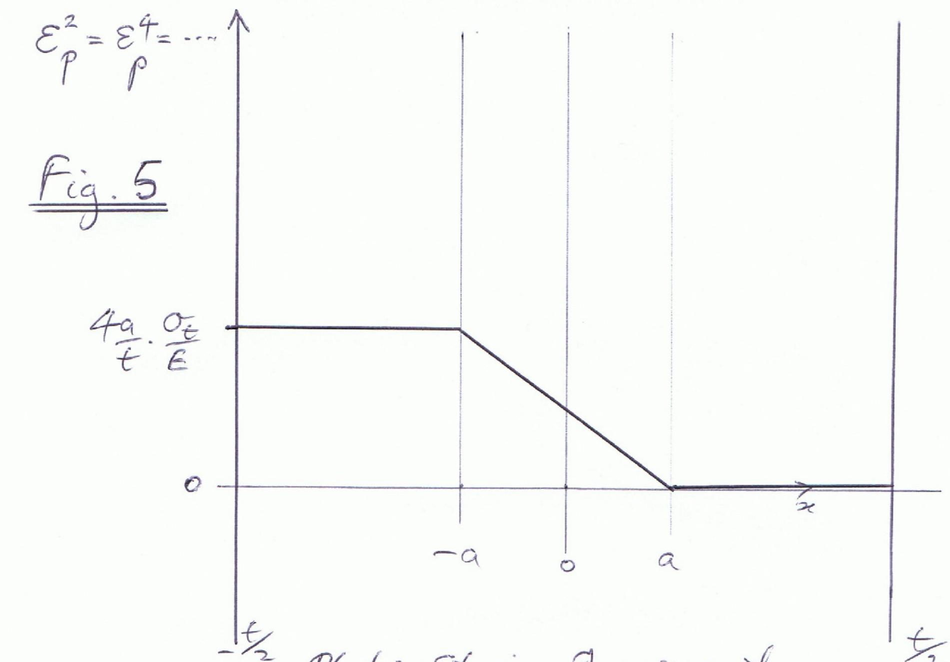


Fig. 5

Plastic Strain Increments per Half-Cycle

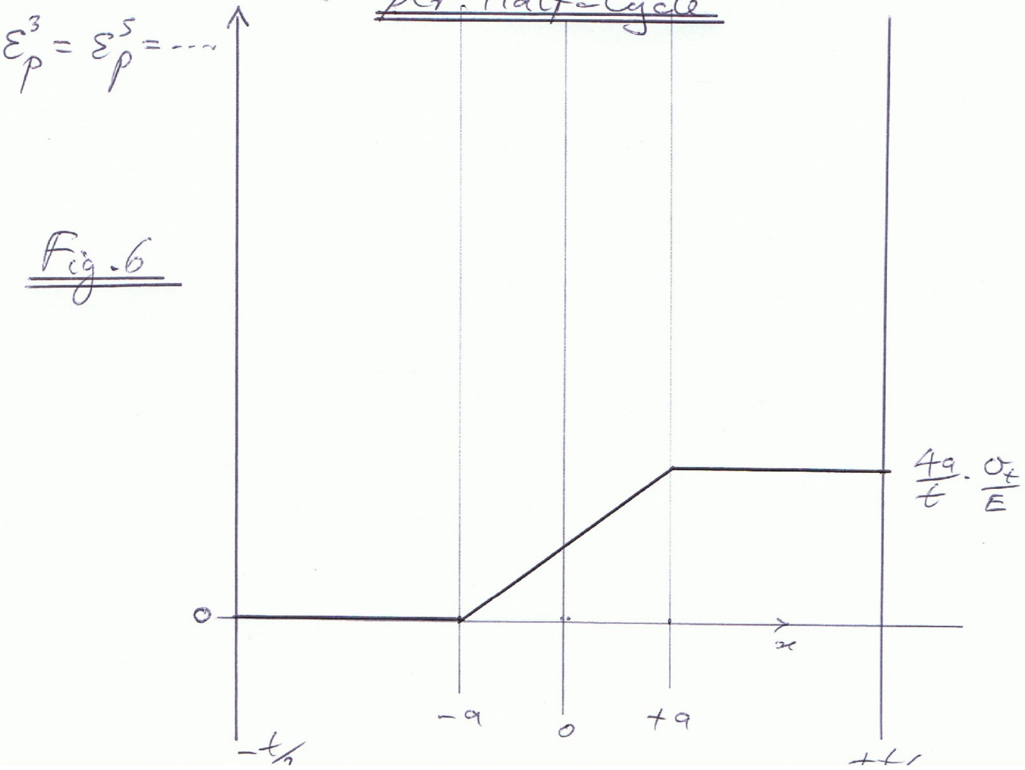


Fig. 6

Fig. 7a
1st half-cycle

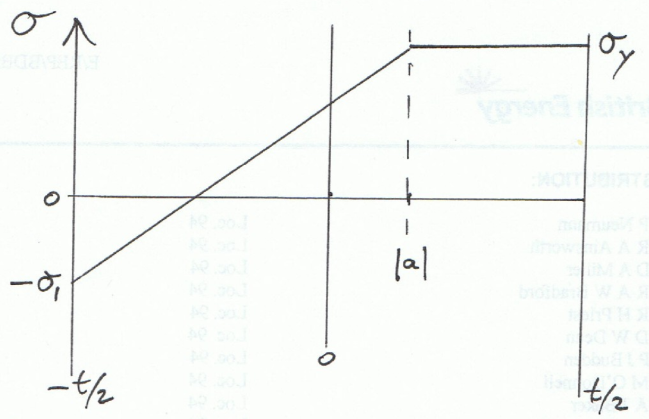


Fig. 7b
1st half-cycle

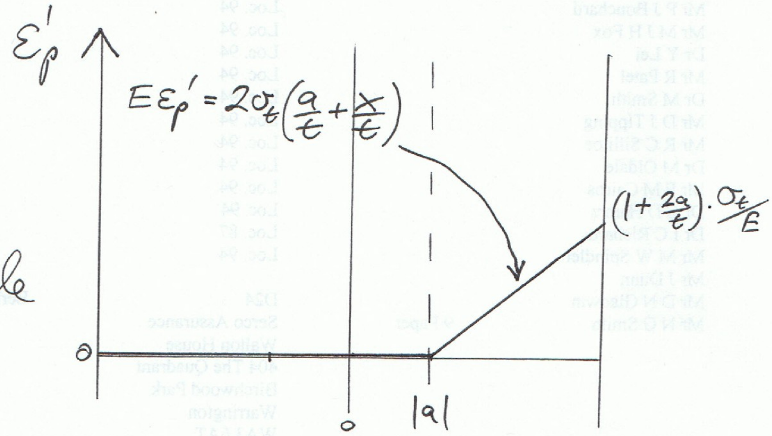
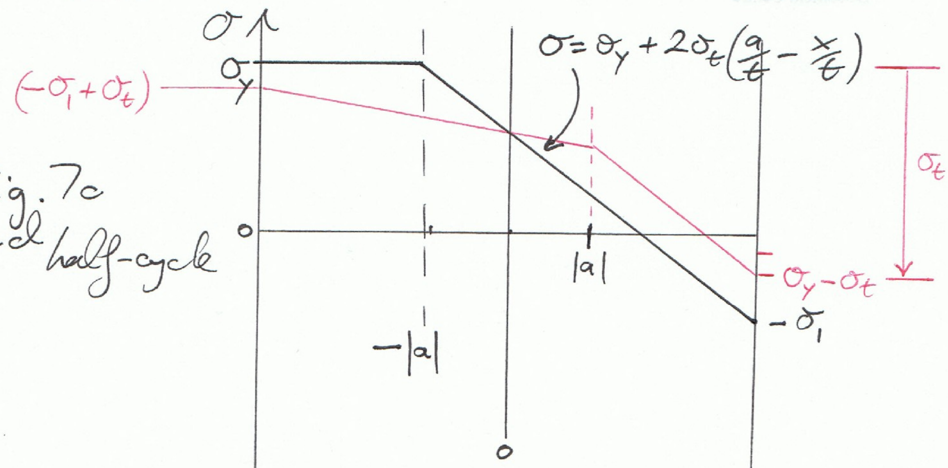


Fig. 7c
2nd half-cycle



Red line = elastic unloading

<u>What happens to Case (A) if $a < 0$?</u>

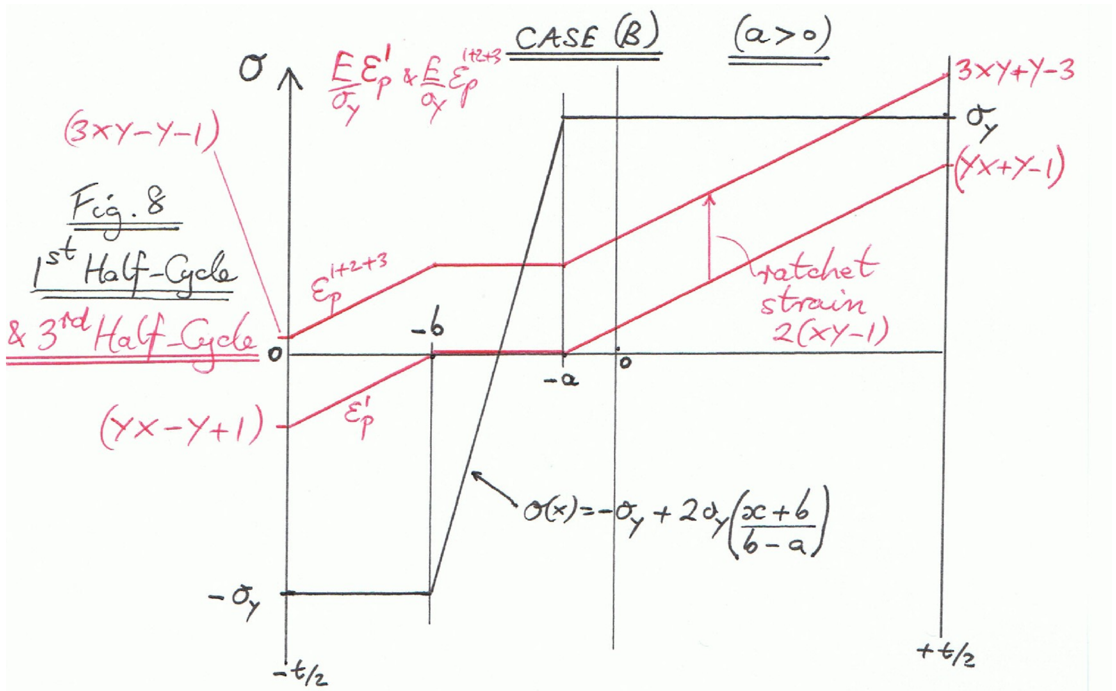


Fig. 8
1st Half-Cycle
& 3rd Half-Cycle

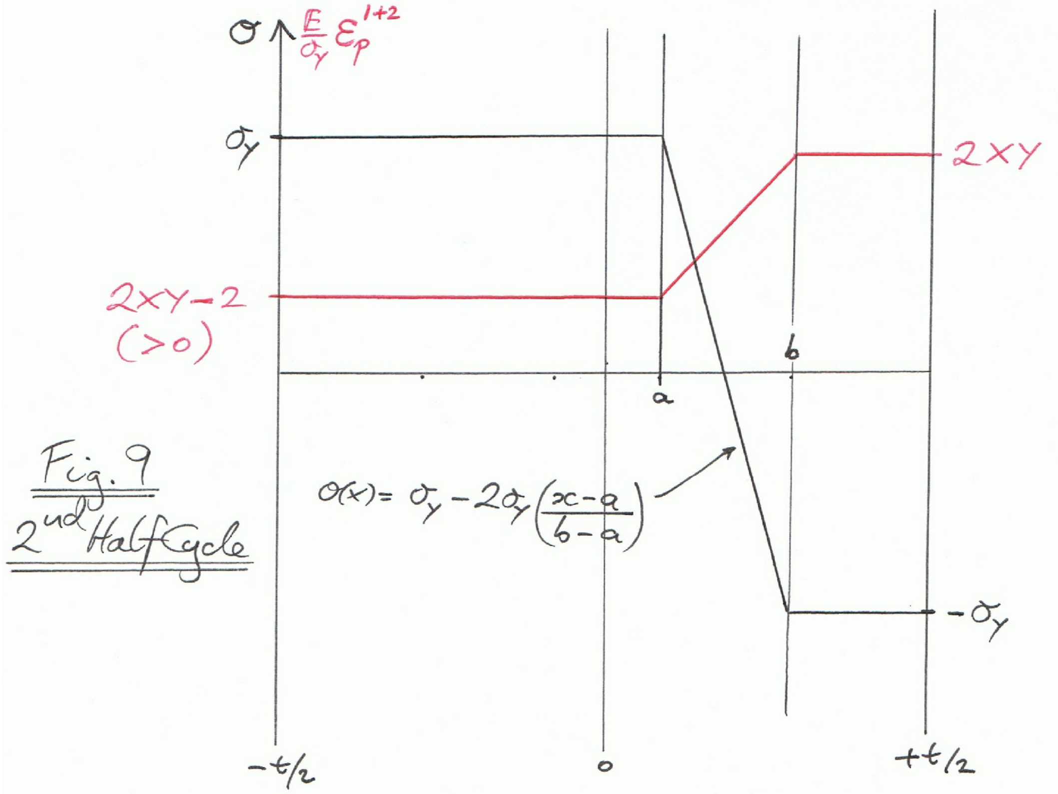
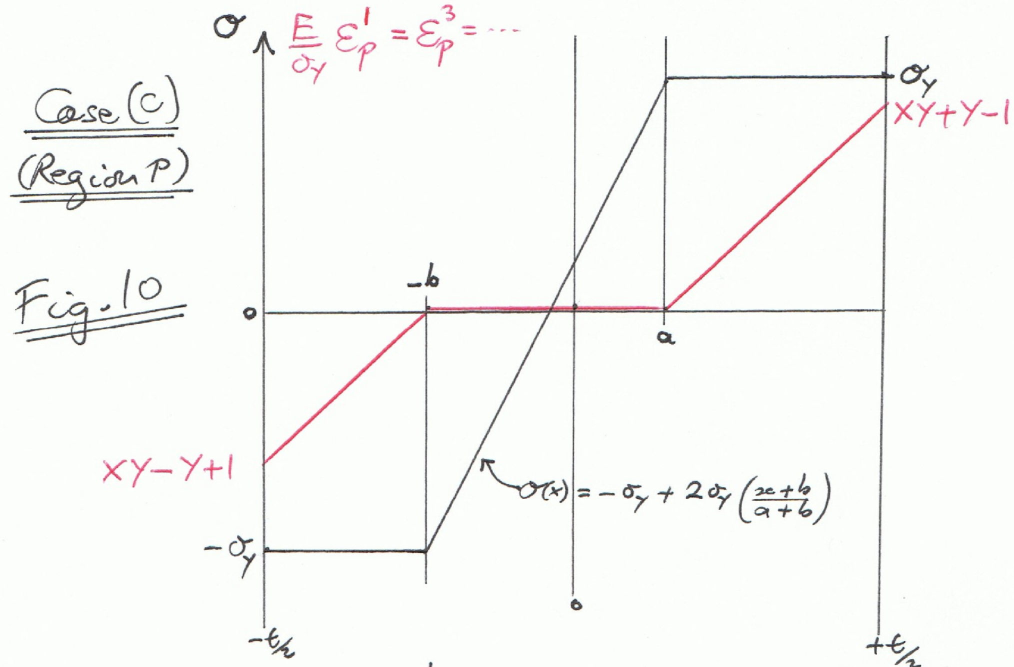


Fig. 9
2nd Half Cycle



Case (c)
(Region P)
Fig. 10
σ
$\frac{E}{\sigma_Y}\varepsilon_p^1 = \varepsilon_p^3 = \cdots$
σ_Y
XY+Y−1
−b
o
a
XY−Y+1
$\sigma(x) = -\sigma_Y + 2\sigma_Y\left(\frac{x+b}{a+b}\right)$
$-\sigma_Y$
o
$-t/2$
$+t/2$

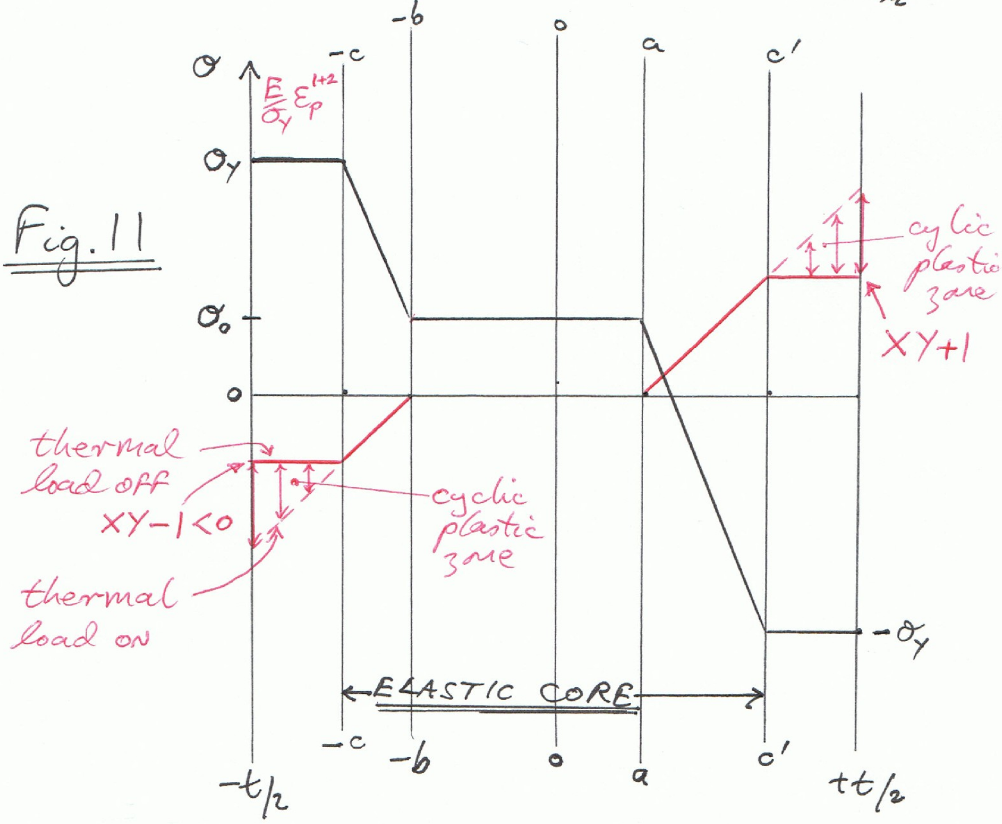
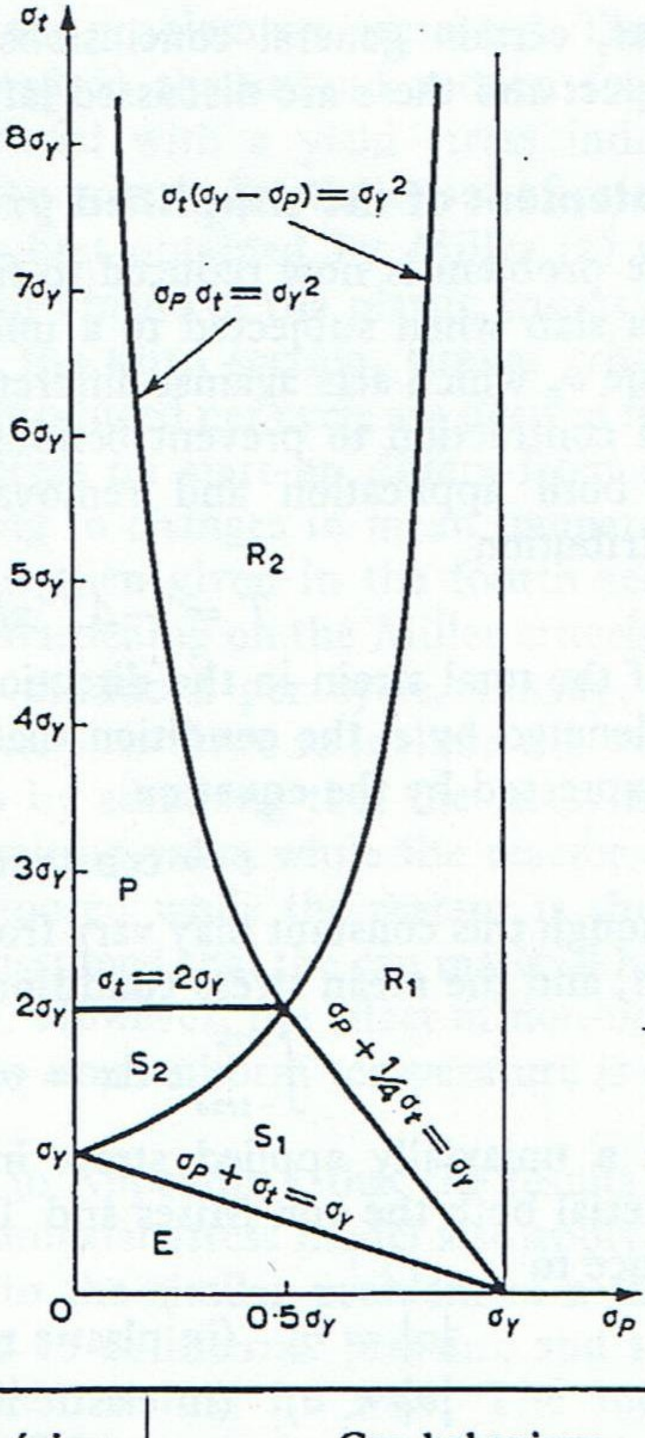


Fig. 11
σ
$\frac{E}{\sigma_Y}\varepsilon_p^{1+2}$
−b
o
a
−c
c′
σ_Y
σ_0
o
cyclic plastic zone
XY+1
thermal load off
XY−1<0
cyclic plastic zone
thermal load on
$-\sigma_Y$
ELASTIC CORE
−c
−b
o
a
c′
$-t/2$
$+t/2$

Figure 12 The Bree Diagram



Stress régime	Can behaviour
R_1 and R_2	Ratchetting
S_1 and S_2	Shakedown after first half-cycle
P	Plastic cycling
E	Elastic