Appendix A2 - Derivation of the Neutron-Proton Capture Cross-Sections In This Universe

1. Introduction

In this Appendix we derive the cross-section for neutron capture on a proton (or vice-versa) and the closely related cross-section for photodisintegration of a deuteron into a neutron and a proton. These reactions may be written $p + n \leftrightarrow D + \gamma$. The derivations are carried out so as to demonstrate that the resulting formulae are equally applicable in an alternative universe in which the strong nuclear coupling constant, g_s, has a different magnitude.

The method is based on evaluating explicitly the relevant Schrodinger wavefunctions, for both free and bound states. Matrix elements of appropriate interaction Hamiltonians are calculated between such states. Fermi's Golden Rule is then used to translate the matrix element into a cross-section. Cross-sections are derived for both a magnetic dipole interaction, in which the magnetic field of a photon couples with the magnetic moment of a deuteron, and also for an electric dipole interaction, in which the electric field of the photon couples with the charge of the proton.

The resulting cross-sections are compared with standard formulae from the literature. Numerical results are also given. The magnetic dipole cross-section is generally dominant.

The effect of a change in the strength of the nuclear force is manifest in the cross-section equations via the deuteron binding energy, B. Some caution needs to be exercised when employing these formulae since, if g_s is increased sufficiently (that is, by more than ~10%), the deuteron will have a stable singlet state as well as the usual triplet state (as shown in the Critique of the Cosmic Coincidences, Chapter 9B). Hence, the cross-sections associated with the singlet deuteron state would also be required for such hypothetical universes. The total capture cross section in these universes, for unpolarised nucleons, would be the sum of the singlet and triplet cross-sections.

In cases for which g_s is increased sufficiently (namely by more than ~10%), the proton-proton capture cross section can be evaluated using essentially the same methods, as can the diproton photodisintegration cross-section. This will be dealt with in Appendix A4. By way of a preview, it will be shown in Appendix A4 that, because we are dealing with identical particles in this case, the p-p capture cross-section is many orders of magnitude less than that for n-p capture. This is because the dominant contribution to n-p capture, via the coupling of the magnetic field to the spin, is zero for identical particles. Moreover, even the far smaller electric dipole contribution to n-p capture is zero to first order in perturbation theory for the p-p system. Hence, p-p capture proceeds only via a quadrupole interaction, which is much weaker than the dipole interactions. The demonstration that the diproton production rate is slow is our ultimate objective. The present Appendix serves primarily as a confirmation that our methodology provides reliable results in this universe.

2. The n-p System: The Schrodinger Solutions and Their Properties

The cross sections will depend upon matrix elements between wavefunctions representing the initial and final states of the two nucleons. We shall be concerned exclusively with non-relativistic nucleons of just a few MeV energy or less. We shall thus use the Schrödinger equation with a square-well nuclear potential, depth V_0 and radial width 'a'. For the n-p system there is no Coulomb interaction, so the Schrödinger equation is simply,

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V\right]\psi = E\psi, \qquad \text{where } V = V_0 \quad \text{for } r \le a, \quad \text{else } V = 0 \tag{1}$$

In Equ.(1), the wavefunction $\psi(\bar{r})$ represents a neutron-proton pair, with $\bar{r}$ being the relative position vector, and m is their reduced mass, i.e. $m = M_n M_p / (M_n + M_p)$. The energy E is the total energy of the two nucleons (excluding their rest mass) in the centre of mass system, i.e. twice the energy of each nucleon (in the approximation that $M_p \approx M_n$).

We shall need only bound S states and free S and P states. In the actual universe, the only bound deuteron state is indeed the S state. When we consider a stronger nuclear force, the binding energy of this triplet S state will increase. If it is increased enough, the first new bound state to appear will be the singlet S state. It can be shown that the triplet P (L = 1) state does not arise until the nuclear force is considerably stronger than that necessary to bind the singlet S state. An even greater nuclear force is required to bind the singlet P state. Thus, as far as bound states are concerned, we are only interested in S states.

However, a free neutron/proton pair can combine to form a deuteron from an initial S or P state. Conversely, a deuteron can photodisintegrate into a free neutron-proton pair in either an S or a P state. Thus, we need to examine both the S and P states of free nucleons.

2.1 Neutron-Proton S States

2.1.1 Bound Neutron-Proton S States

The relevant solution to Equ.(1) may be shown by substitution to be,

$$\psi = a_i \frac{\sin\rho_i}{\rho_i}, \quad \rho_i = \alpha r, \quad \alpha = \sqrt{2m(V_0 - B)}/\hbar, \quad \text{for } r \le a \tag{2}$$

$$\psi = a_e \frac{e^{-\rho_e}}{\rho_e}, \quad \rho_e = \beta r, \quad \beta = \sqrt{2mB}/\hbar, \quad \text{for } r > a \tag{3}$$

where B is the binding energy (i.e. this solution has energy eigenvalue E = -B, so that E < 0 but B > 0). The coefficients α and β are defined so as to be real. The subscripts $_i$ and $_e$ represent 'inside' and 'outside' in obvious fashion. Note that the functions appearing in Equs.(1,2) are the spherical Bessel function of order zero $j_o(\rho_i)$ and the

associated Hankel function of the first kind, $h_0^{(1)}(i\rho_e)$. The Neumann function n_0 is excluded from the interior solution by the requirement that the wavefunction be finite at the origin. Similarly, the Hankel function of the second kind is excluded from the exterior solution since it involves a positive exponent, and hence is not normalisable.

The coefficients a_i and a_e appearing in Equs.(2,3) are found, together with the binding energy quantisation condition, by requiring that the wavefunction and its first derivative are continuous at r = a, plus the requirement that the wavefunction be normalised. Continuity of the wavefunction immediately leads to,

$$a_i \frac{\sin\xi}{\xi} = a_e \frac{e^{-\eta}}{\eta}, \qquad \text{where, } \xi = \alpha a \quad \text{and} \quad \eta = \beta a \tag{4}$$

Continuity of the gradient of the wavefunction is easily shown to lead to the quantised binding energy condition,

$$\eta = -\xi \cot\xi \tag{5}$$

From the definition, Equ.(4), for a given potential well the solution is constrained to lie on a circle in $\xi - \eta$ space, i.e.,

$$\xi^2 + \eta^2 = \kappa_o^2 \qquad \text{where} \quad \kappa_0 = a\sqrt{2mV_0}/\hbar \tag{6}$$

Thus, Equs.(5,6) together define discrete possibilities for the binding energy, B. If $\kappa_0 < \pi/2$ there is no bound state. If $\pi/2 < \kappa_0 < 3\pi/2$ there is exactly one bound state, whereas for $3\pi/2 < \kappa_0 < 5\pi/2$ there are exactly two bound states, etc[1].

We can now find the coefficients a_i and a_e explicitly by requiring that the wavefunction be normalised, i.e. requiring that,

$$\int_{\text{all } E^3} |\psi|^2 dV = 1 \tag{7}$$

The contributions to the integral are evaluated separately for the interior (r < a) and exterior (r > a) parts, giving,

$$\int_{r<a} |\psi|^2 dV = \int_0^a 4\pi r^2 dr \cdot |a_i|^2 \frac{\sin^2 \alpha r}{(\alpha r)^2} = \frac{2\pi |a_i|^2}{\alpha^2}\left(a - \frac{\sin 2\xi}{2\alpha}\right) \tag{8}$$

$$\int_{r>a} |\psi|^2 dV = \int_a^\infty 4\pi r^2 dr \cdot |a_e|^2 \frac{e^{-2\beta r}}{(\beta r)^2} = \frac{2\pi |a_e|^2}{\beta^3} e^{-2\beta a} = \frac{2\pi |a_e|^2 a^3}{\eta^3} e^{-2\eta} \tag{9}$$

[1] That is, counting only S waves of the same net spin. The states enumerated correspond to different principal quantum numbers, n.

Note that the dimensions of α,β are 1/length, and that of the squared-coefficients a_i^2 and a_e^2 is 1/volume. The coefficient a_e may be eliminated from Equ.(9) in favour of a_i with the aid of Equ.(4). The sum of (8) and (9) may be simplified with the help of the identity,

$$\frac{\sin^2\xi}{\beta}-\frac{\sin 2\xi}{2\alpha}=\frac{a}{\eta} \tag{10}$$

from which it follows from (7) that,

$$\left|a_i\right|^2=\frac{\alpha^2}{2\pi a\left(1+1/\eta\right)} \tag{11}$$

from which the external coefficient a_e may also be found using Equ.(4) if required. Incidentally, we note that Equ.(4) shows that the two coefficients a_i and a_e are in phase, and hence may both be taken as real, so we can drop the modulus sign.

2.1.2 Orthogonality of Bound States of Different Energy

For a given potential function, V(r), there is a general theorem that any two solutions of Schrodinger's equation with different energies are orthogonal (even if they have the same angular dependence). This orthogonality is important to arguments which will be deployed for the proton-proton system later. Hence we carry out a check that the solution of Equs.(2,3) respects orthogonality. Thus, we intend to show that,

$$\int_{\text{all }E^3}\psi_1^*\psi_2 dV=0 \tag{12}$$

where the subscripts denote solutions with different binding energies. The internal part of this integral is,

$$\begin{aligned}\Im_i&=\int_0^a 4\pi r^2 dr\cdot a_{1i}a_{2i}\frac{\sin\alpha_1 r}{\alpha_1 r}\cdot\frac{\sin\alpha_2 r}{\alpha_2 r}\\&=\frac{2\pi a_{1i}a_{2i}}{\alpha_1\alpha_2}\left\{\frac{\sin\left(\alpha_1-\alpha_2\right)a}{\left(\alpha_1-\alpha_2\right)}-\frac{\sin\left(\alpha_1+\alpha_2\right)a}{\left(\alpha_1+\alpha_2\right)}\right\}\end{aligned} \tag{13}$$

and the external part is,

$$\begin{aligned}\Im_e&=\int_a^\infty 4\pi r^2 dr\cdot a_{1e}a_{2e}\frac{e^{-\beta_1 r}}{\beta_1 r}\cdot\frac{e^{-\beta_2 r}}{\beta_2 r}\\&=\frac{4\pi a_{1e}a_{2e}e^{-(\beta_1+\beta_2)a}}{\beta_1\beta_2\left(\beta_1+\beta_2\right)}\\&=\frac{4\pi a^2 a_{1i}a_{2i}\sin\xi_1\sin\xi_2}{\xi_1\xi_2\left(\beta_1+\beta_2\right)}\end{aligned} \tag{14}$$

(the latter step uses Equ.4). Equ.(13) is easily simplified to give,

$$\Im_i = \frac{4\pi a_{1i} a_{2i}}{\alpha_1 \alpha_2 \left(\alpha_1^2 - \alpha_2^2\right)} \left[\alpha_2 \sin\xi_1 \cos\xi_2 - \alpha_1 \sin\xi_2 \cos\xi_1\right] \tag{15}$$

Using Equ.(5) to write $\cos\xi = -(\eta/\xi)\sin\xi$, Equ.(15) becomes,

$$\Im_i = \frac{4\pi a^2 a_{1i} a_{2i} \left(\beta_1 - \beta_2\right)}{\left(\alpha_1^2 - \alpha_2^2\right)} \left[\frac{\sin\xi_1 \sin\xi_2}{\xi_1 \xi_2}\right] \tag{16}$$

However we have that,

$$\beta_1^2 - \beta_2^2 = \frac{2m}{\hbar^2}\left(B_1 - B_2\right) \tag{17a}$$

and

$$\alpha_1^2 - \alpha_2^2 = \frac{2m}{\hbar^2}\left(E_1 - E_2\right) = \frac{2m}{\hbar^2}\left(-B_1 + B_2\right) = -\left(\beta_1^2 - \beta_2^2\right) \tag{17b}$$

Hence, from Equs.(14) and (16) we see that $\Im_i = -\Im_e$, in other words the total integral of Equ.(12), $\Im_i + \Im_e$, is zero. Thus, the two bound-state wavefunctions are orthogonal as claimed. QED.

2.1.3 The Zero-Range Approximation For Bound States

It will be useful later in simplifying the algebra to develop a "zero-range" approximation in which $a \to 0$ and $V_0 \to \infty$ in such a manner as to ensure that the binding energy, B, remains constant. Contrary to what one might initially guess, this does <u>not</u> mean holding the combination $a^2 V_0$ constant. If this were done, the dimensionless parameters ξ and η would remain constant, which would lead to $\beta = \eta/a$ diverging as 1/a and hence B diverging as $1/a^2$. Instead, it is clear that η must reduce towards zero as 'a' does, so that $\beta = \eta/a$ remains constant, and hence B also remains constant. By virtue of the quantisation relation, Equ.(5), it follows that ξ must approach $\pi/2$. (We confine attention to the lowest lying bound state. We shall not be considering higher lying bound states of the same spin to arise. They would exist only if g_s were increased by a factor of 3). We check that this "zero-range" approximation continues to respect the normalisation of the wavefunction as follows:-

By using $\xi = \pi/2$ and $\alpha = \pi/2a$ in Equ.(8) we get the interior integral to be,

$$\aleph_i = \int_{r<a} |\psi|^2 dV \to \frac{8}{\pi} a^3 a_i^2 \tag{18}$$

Noting that Equ.(9) may be written in terms of the interior coefficient, using Equ.(4), and substituting $\xi = \pi/2$ and $\alpha = \pi/2a$ gives,

$$\aleph_e = \int_{r>a} |\psi|^2 dV = \frac{2\pi |a_i|^2 \sin^2\xi}{\alpha^2 \beta} \to \frac{8a^2 a_i^2}{\pi\beta} \tag{19}$$

Thus, the ratio of the interior to exterior integrals $\aleph_i / \aleph_e$ is of order 'aβ ' and hence tends to zero as 'a' does. Thus, in the zero-range limit, the nucleons tend to lie entirely outside the range of the nuclear force. The exterior integral alone must therefore tend to unity, $\aleph_e \to 1$, which gives,

$$a_i^2 \to \frac{\pi\beta}{8a^2} \tag{20}$$

The same result is obtained by considering the limit of Equ.(11). Finally, Equ.(4) is used to give the exterior coefficient in the zero range approximation, using $\xi = \pi/2$ and $\eta = \beta a \to 0$, giving,

$$a_e^2 \to \frac{\beta^3}{2\pi} \tag{21}$$

In fact, we have already found the coefficients for the bound ^{3}S state without recourse to the zero-range approximation, i.e. Equ.(11). It will be required later to know the zero-range approximation for the coefficients to second order. The second order approximations for α and ξ will be discussed in Section 2.2.2 and are given in Equ.(31). The corresponding approximation for $\sin\xi/\xi$ is given by Equ.(64a), and will be needed in Equ.(4). Thus, we find the second order approximations for the two coefficients using Equs.(11) and (4) to be,

$$a_i^2 = \frac{\pi\beta}{8a^2}(1-\zeta\beta a), \quad a_e^2 = \frac{\beta^3}{2\pi}\left(1+\frac{8}{\pi^2}\beta a\right), \quad \zeta = 1 - \frac{8}{\pi^2} \tag{20b,21b}$$

2.2.1 Free Neutron-Proton S States

We now consider the positive energy S-state solutions of Equ.(1). These may be checked by substitution to be,

$$\psi = a_i \frac{\sin\rho_i}{\rho_i}, \qquad \rho_i = \alpha r, \quad \alpha = \sqrt{2m(V_0 + E)}/\hbar, \quad \text{for } r \le a \tag{22}$$

$$\psi = a_{es}\frac{\sin\rho_e}{\rho_e} + a_{ec}\frac{\cos\rho_e}{\rho_e}, \qquad \rho_e = kr, \quad k = \sqrt{2mE}/\hbar, \quad \text{for } r > a \tag{23}$$

Note that we have used the same symbols ρ_i and α for the interior solution as for the bound state, since the expressions are algebraically identical. However, the energy, E, is positive here rather than negative. Thus, the numerical values of α cannot be the same. The exterior solution is now oscillatory (i.e. wave-like) and consists of two terms – since both tend to zero at infinity. The three coefficients in Equs.(22, 23) can be found, in principle, using the requirements of continuity of the wavefunction and its first derivative at r = a, plus the normalisation condition. The continuity conditions are,

$$a_i \frac{\sin\xi}{\xi} = a_{es} \frac{\sin\kappa}{\kappa} + a_{ec} \frac{\cos\kappa}{\kappa} \qquad \text{where} \quad \xi = \alpha a \quad \text{and} \quad \kappa = ka \tag{24}$$

$$\alpha a_i \left\{ \frac{\cos\xi}{\xi} - \frac{\sin\xi}{\xi^2} \right\} = ka_{es} \left\{ \frac{\cos\kappa}{\kappa} - \frac{\sin\kappa}{\kappa^2} \right\} - ka_{ec} \left\{ \frac{\sin\kappa}{\kappa} + \frac{\cos\kappa}{\kappa^2} \right\} \tag{25}$$

Thus, in principle, the two exterior coefficients may be found in terms of the interior coefficient from Equs.(24, 25). Normalisation is now considered. The interior integral is algebraically identical to Equ.(8), i.e.,

$$\aleph_i = \int_{r<a} |\psi|^2 dV = \frac{2\pi |a_i|^2}{\alpha^2} \left(a - \frac{\sin 2\xi}{2\alpha} \right) \tag{26}$$

The exterior integral is,

$$\aleph_e = \int_a^{r_{max}} 4\pi r^2 dr \left[a_{es} \frac{\sin kr}{kr} + a_{ec} \frac{\cos kr}{kr} \right]^2 \tag{27}$$

Note that the domain of the exterior integral is finite, up to some large r_{max}. This is necessary in order to make the integral (27) finite. It means that the coefficients of the free wavefunction can only be found in terms of this arbitrary normalisation dimension. This is always the case with free (wave-like) states. Ultimately this arbitrary normalisation dimension will cancel out from any expression for a physically measurable quantity (usually by cancellation against the density of states term). Whilst r_{max} is arbitrary, it must be very large compared with all the characteristic dimensions of the problem (otherwise confining the state to a small finite volume would have physically measurable consequences). Thus we require,

$$r_{max} >> a \quad \text{and} \quad r_{max} >> 1/k \quad \text{and} \quad r_{max} >> 1/\alpha \quad \text{and} \quad r_{max} >> 1/\beta \quad \text{and} \quad r_{max} >> 1/k_0$$

where k_0 in the last inequality is $\sqrt{2mV_0}/\hbar$. As an illustration consider the deuteron. We have a ~ 2fm, $1/k_0$ ~ 1fm (V_0 ~ 37 MeV), $1/\beta$ ~ 4.3fm (B = 2.22MeV). If we consider nucleon kinetic energies of (say) 1MeV, 0.1Mev and 0.01MeV then 1/k is respectively 6.4fm, 20fm and 64fm. The size scale $1/\alpha$ is virtually the same as $1/k_0$ in the energy range of interest, and is not important anyway since 'a' sets the scale for the interior solution. Thus, in the energy range of interest it is 1/k which forms the largest size scale associated with the nucleons, and our normalisation region must be chosen with r_{max} >> 1/k. There is one further size scale, namely the wavelength of a photon with energy sufficient to disintegrate the deuteron, i.e. λ < hc/2.22MeV = 89fm.

Returning to the exterior integral of Equ.(27), the exact integration yields an expression which involves the difference between trigonometric functions evaluated at $2\rho_{e,max} = 2kr_{max}$ and $2\kappa = 2ka$. Considerable simplification results if we choose these two parameters to differ by an integral multiple of 2π, ie.,

$$kr_{max} = ka + n\pi \quad \text{where } n = \text{integer} \tag{28}$$

(of course, n will need to be a very large integer because $r_{max} >> a$). Equ.(27) then becomes,

$$\aleph_e = \frac{2\pi}{k^2}\left(a_{es}^2 + a_{ec}^2\right)\left(r_{max} - a\right) \tag{29}$$

Thus, in principle, having found the two exterior coefficients in terms of the interior coefficient using Equs.(24, 25), the latter is found explicitly by adding (26) and (29) and equating to unity. The algebra appears rather forbidding. Instead we consider this procedure only in the zero-range approximation in the next Section.

2.2.2 The Zero-Range Approximation For Free S States

The zero-range approximation was introduced in Section 2.1.3. We use the same approximation here except that we now need to retain higher order terms. This is necessary to ensure that orthogonal states remain orthogonal in the approximation used. Thus, proceeding immediately to the limiting values $\xi \to \pi/2$ and $\eta \to 0$, for example, is too crude. We require expressions which retain terms of order 'a'.

Despite the fact that we are dealing here with free states, recall that the "zero-range" approximation is only defined with respect to a specified binding energy, B. This is because the limiting process $a \to 0, V_0 \to \infty$ is defined by the requirement that B be constant. Hence, derivation of the order 'a' term in ξ requires consideration of the bound-state quantisation condition $\eta = -\xi\cot\xi$. The value of η to first order is, trivially, $\eta = \beta a$. The derivative of the quantisation condition gives the size of the variation in ξ for a given variation in η, thus,

$$d\eta = -d\xi.\cot\xi + \xi\,\mathrm{cosec}^2\xi.d\xi \tag{30}$$

Thus, near the limiting value $\eta = 0, \xi = \pi/2$ the cot term is zero and the cosec term is unity, and the small change in η is just η itself, giving,

$$d\xi = \frac{2}{\pi}\eta \text{ and hence } \xi \approx \frac{\pi}{2} + \frac{2}{\pi}\beta a \text{ and } \alpha \approx \frac{\pi}{2a} + \frac{2}{\pi}\beta \tag{31}$$

We now use (31) to approximate the LHS of Equs.(24,25), and also retain only the first terms of the trig functions on the RHS, giving,

$$\frac{2}{\pi}\left(1 - \frac{4}{\pi^2}\beta a\right)a_i = a_{es} + \left(\frac{1}{\kappa} - \frac{\kappa}{2}\right)a_{ec} \tag{32a}$$

$$\frac{4}{\pi^2}\left(1 + \zeta\beta a\right)\alpha a_i = \frac{k\kappa}{3}a_{es} + k\left(\frac{1}{\kappa^2} + \frac{1}{2}\right)a_{ec} \tag{32b}$$

where, $\zeta = 1 - 8/\pi^2$. Expressing the RHS of Equs.(32a,b) as a matrix, the inverse of the matrix is (retaining leading terms in κ in each component),

$$\begin{bmatrix} 1+\kappa^2/3 & -\kappa/k \\ -\kappa^3/3 & \kappa^2/k \end{bmatrix} \tag{33}$$

Multiplying the above matrix by the column vector formed by the LHS of Equs.(32a,b) thus gives, retaining only leading terms in 'a',

$$a_{es} \to -\frac{2}{\pi}\beta a a_i \tag{34a}$$

$$a_{ec} \to \frac{2}{\pi}ka a_i \tag{34b}$$

We can now find the normalisation integrals explicitly. Using Equs.(34a,b) and $\xi = \pi/2$ and $\alpha = \pi/2a$, Equ.(26) gives the interior integral as,

$$\aleph_i = \frac{8}{\pi}a^3 a_i^2 \tag{35}$$

which is the same as the interior integral for bound states in the zero-range approximation, i.e. Equ.(18). However, Equ.(29) gives the exterior integral as,

$$\aleph_e = \frac{8}{\pi}\left(1+\frac{\beta^2}{k^2}\right)r_{max}a^2 a_i^2 = \frac{8}{\pi}\left(1+\frac{B}{E}\right)r_{max}a^2 a_i^2 \tag{36}$$

and hence differs from that for the bound state, Equ.(19), which does not involve a finite normalisation radius, r_{max}. As before we see that the ratio of the integrals is $\aleph_i/\aleph_e \approx O(a) \to 0$. Hence, the normalisation condition is $\aleph_e \to 1$ which gives,

$$a_i^2 = \frac{\pi}{8(1+B/E)r_{max}a^2} \tag{37}$$

from which the exterior coefficients may be found using Equs.(34a,b). Thus, as we found also for the bound states, in the zero-range limit, the nucleons tend to lie entirely outside the range of the nuclear force.

2.2.3 Orthogonality of Free and Bound States: Zero-Range Approximation

The zero-range approximation has been introduced to facilitate calculation of the matrix elements which are needed to compute the n-p system cross sections. It is therefore essential that states which are truly orthogonal also have $<\psi_1|\psi_2>=0$ when evaluated using the zero-range approximation. Were this not the case we could have little faith in the required capture/disintegration matrix elements evaluated using this same approximation. In this Section we shall check that the zero-range approximation does indeed yield $<\psi_1|\psi_2>=0$ between a bound state (ψ_1) and a free state (ψ_2). The interior integral is identical to Equ.(13), i.e.,

$$\begin{aligned}\Im_i &= \int_0^a 4\pi r^2 dr \cdot a_{1i} a_{2i} \frac{\sin\alpha_1 r}{\alpha_1 r} \cdot \frac{\sin\alpha_2 r}{\alpha_2 r} \\ &= \frac{2\pi a_{1i} a_{2i}}{\alpha_1\alpha_2}\left\{\frac{\sin(\alpha_1-\alpha_2)a}{(\alpha_1-\alpha_2)} - \frac{\sin(\alpha_1+\alpha_2)a}{(\alpha_1+\alpha_2)}\right\}\end{aligned} \tag{38}$$

Substitution of the zero-range estimate of the coefficient a_{1i} from Equ.(20), together with $\alpha_2 \to \alpha_1 \to \pi/2a$ gives,

$$\Im_i = 2\pi a_{2i}\left(\frac{2a}{\pi}\right)^2\left(\frac{\pi\beta}{8a^2}\right)^{1/2}\left\{a - \frac{a}{\pi}(\sin\pi = 0)\right\} = \left(\frac{8\beta}{\pi}\right)^{1/2} a^2 a_{2i} \to 0 \tag{39}$$

The exterior integral consists of two parts,

$$\Im_{es} = \int_a^{r_{max}} a_{1e}\frac{e^{-\beta r}}{\beta r} a_{es}\frac{\sin kr}{kr}\cdot 4\pi r^2 dr = \frac{4\pi a_{1e}a_{es}}{\beta k}\chi_s \tag{40}$$

and,
$$\Im_{ec} = \int_a^{r_{max}} a_{1e}\frac{e^{-\beta r}}{\beta r} a_{ec}\frac{\cos kr}{kr}\cdot 4\pi r^2 dr = \frac{4\pi a_{1e}a_{ec}}{\beta k}\chi_c \tag{41}$$

where,
$$\chi_s = \int_a^{r_{max}} e^{-\beta r}\sin kr\cdot dr = \left(\frac{\beta\sin ka + k\cos ka}{\beta^2+k^2}\right)e^{-\beta a} \tag{42}$$

and,
$$\chi_c = \int_a^{r_{max}} e^{-\beta r}\cos kr\cdot dr = \left(\frac{\beta\cos ka - k\sin ka}{\beta^2+k^2}\right)e^{-\beta a} \tag{43}$$

and where the integrals (42,43) have been evaluated by assuming the limit $r_{max} \to \infty$. In the zero-range approximation ($a \to 0$) these become,

$$\chi_s = \left(\frac{k}{\beta^2+k^2}\right) \quad \text{and} \quad \chi_c = \left(\frac{\beta}{\beta^2+k^2}\right) \tag{44}$$

Thus, substituting Equs.(44) into Equs.(40,41), together with Equs.(34a,b) for the exterior free state coefficients, gives,

$$\Im_{es} = \frac{4\pi a_{1e}}{\beta k}\left(-\frac{2}{\pi}\beta a\right)a_{2i}\frac{k}{\beta^2+k^2} = -8a_{1e}a_{2i}\frac{a}{\beta^2+k^2} \tag{45}$$

and,
$$\Im_{ec} = \frac{4\pi a_{1e}}{\beta k}\left(\frac{2}{\pi}ka\right)a_{2i}\frac{\beta}{\beta^2+k^2} = 8a_{1e}a_{2i}\frac{a}{\beta^2+k^2} = -\Im_{es} \tag{46}$$

Thus, in view of the interior integral once again tending to zero in the zero-range approximation, we have $<\psi_1|\psi_2> = \Im_{es} + \Im_{ec} = 0$ by virtue of Equs.(45,46). QED.

2.3 Neutron-Proton Bound-Free Matrix Element: S States

In this Section we use the results of the previous Sections to find the matrix element $< \psi_{bound}(^3S) | \psi_{free}(^1S) >$ in the zero-range approximation. Note that if the bound and free states were in the same spin state (i.e. both triplet or both singlet), then this matrix element would be zero, as shown in Section 2.2.3, above. The reason why orthogonality does not apply when the spin states are different is that the nuclear potential differs for the two spin states. Thus, the free and bound states are solutions of *different* Schrödinger equations. We may summarise the results of the preceding sections for the wavefunctions as follows,

$$\psi_{bound}(^3S) = a_i \frac{\sin\rho_i}{\rho_i}, \quad \rho_i = \alpha r, \quad \alpha = \sqrt{2m(V_0 - B)}/\hbar, \quad \text{for } r \le a \tag{2}$$

$$\psi_{bound}(^3S) = a_e \frac{e^{-\rho_e}}{\rho_e}, \quad \rho_e = \beta r, \quad \beta = \sqrt{2mB}/\hbar, \quad \text{for } r > a \tag{3}$$

where, V_0 represents the triplet nuclear potential and B the corresponding binding energy. In the zero-range approximation we have,

$$a_i^2 \to \frac{\pi\beta}{8a^2} \qquad a_e^2 \to \frac{\beta^3}{2\pi} \qquad \alpha \to \frac{\pi}{2a} \tag{20, 21}$$

Similarly, for the free singlet state,

$$\psi_{free}(^1S) = \tilde{a}_i \frac{\sin\tilde{\rho}_i}{\tilde{\rho}_i}, \quad \tilde{\rho}_i = \tilde{\alpha} r, \quad \tilde{\alpha} = \sqrt{2m(\tilde{V}_0 + E)}/\hbar, \quad \text{for } r \le a \tag{22b}$$

$$\psi_{free}(^1S) = a_{es} \frac{\sin\tilde{\rho}_e}{\tilde{\rho}_e} + a_{ec} \frac{\cos\tilde{\rho}_e}{\tilde{\rho}_e}, \quad \tilde{\rho}_e = kr, \quad k = \sqrt{2mE}/\hbar, \quad \text{for } r > a \tag{23b}$$

where the tildas indicate that the free singlet state parameters differ from those of the bound triplet state. In this case, $\tilde{V}_0$ represents the singlet nuclear potential, and hence $\tilde{V}_0 < V_0$). In the zero-range approximation we have,

$$a_{es} \to -\frac{2}{\pi}\tilde{\beta} a \tilde{a}_i \qquad \tilde{\beta} = \sqrt{2m\tilde{B}}/\hbar \tag{34c}$$

$$a_{ec} \to \frac{2}{\pi} k a \tilde{a}_i \tag{34d}$$

$$\tilde{a}_i^2 = \frac{\pi}{8\left(1 + \tilde{B}/E\right) r_{max} a^2} \tag{37b}$$

where $\tilde{B}$ represents the binding energy corresponding to the singlet potential $\tilde{V}_0$. This formulation of the zero-range approximation for the free singlet state is fine – provided that a singlet bound state exists (and hence $\tilde{B}$ is defined). Unfortunately, a

bound singlet state does *not* exist in this universe. In this case we may identify the parameter $\tilde{\beta} = \sqrt{2m\tilde{B}}/\hbar$ with the reciprocal of the singlet scattering length, i.e. $\tilde{\beta} = 1/^{1}a$. Because there is no singlet bound state, the singlet scattering length is negative, and hence so is $\tilde{\beta}$. However, $\tilde{B}$ remains positive.

[Aside: That $\tilde{\beta}$ can be identified with the reciprocal of the scattering length may be established as follows. From standard scattering theory (e.g. Evans P.319), in the limit of zero energy the external wavefunction may be approximated as,

$$LIM(k \to 0) : \psi \to cons\tan t\left[1 - \frac{^{1}a}{r}\right] \tag{47}$$

But Equ.(23b) gives the external wavefunction in the zero energy limit as,

$$LIM(k \to 0) : \psi \to a_{es} + \frac{a_{ec}}{kr} \tag{48}$$

and with the help of Equs.(34c,d) this becomes,

$$LIM(k \to 0) : \psi \to -\frac{2}{\pi}\tilde{\beta}a\tilde{a}_{i}\left\{1 - \frac{1}{\tilde{\beta}r}\right\} \tag{49}$$

Equating (47) and (49) yields $\tilde{\beta} = 1/^{1}a$, as claimed].

We can now calculate the required matrix element,

$$< \psi_{bound}(^{3}S)\left|\psi_{free}(^{1}S)\right. >= \Im_{i} + \Im_{e} \tag{50}$$

where,

$$\begin{aligned}\Im_{i} &= \int_{0}^{a} 4\pi r^{2} dr \cdot a_{i}\tilde{a}_{i}\frac{\sin\alpha r}{\alpha r}\cdot\frac{\sin\tilde{\alpha}r}{\tilde{\alpha}r} \\ &= \frac{2\pi a_{i}\tilde{a}_{i}}{\alpha\tilde{\alpha}}\left\{\frac{\sin(\alpha-\tilde{\alpha})a}{(\alpha-\tilde{\alpha})} - \frac{\sin(\alpha+\tilde{\alpha})a}{(\alpha+\tilde{\alpha})}\right\}\end{aligned} \tag{51}$$

Substitution of the zero-range estimates $\tilde{\alpha} \to \alpha \to \pi/2a$ gives,

$$\Im_{i} = 2\pi a_{i}\tilde{a}_{i}\left(\frac{2a}{\pi}\right)^{2}\left\{a - \frac{a}{\pi}(\sin\pi = 0)\right\} = \frac{8}{\pi}a^{3}a_{i}\tilde{a}_{i} \to 0 \tag{52}$$

The interior integral is thus negligible in the zero-range approximation because the interior coefficients diverge only as 1/a, see (20) and (37b).

The exterior integral is,

$$\Im_e = \int_a^{r_{max}} a_e \frac{e^{-\beta r}}{\beta r}\left\{a_{es}\frac{\sin kr}{kr} + a_{ec}\frac{\cos kr}{kr}\right\}4\pi r^2 dr \tag{53}$$

Apart from the actual values of the coefficients, this integral is identical to the sum of Equs.(40) and (41), and may be evaluated in the zero-range approximation using Equs.(44). Thus we get,

$$\Im_e = \frac{4\pi a_e}{\beta k}\left\{a_{es}\left(\frac{k}{k^2+\beta^2}\right) + a_{ec}\left(\frac{\beta}{k^2+\beta^2}\right)\right\} \tag{54}$$

We now substitute the zero-range approximations for the exterior coefficients from Equs.(21, 34c,d) giving,

$$\begin{aligned}\Im_e &= \frac{4\pi}{\beta k}\left(\frac{\beta^3}{2\pi}\right)^{1/2}\left(\frac{2a\tilde{a}_i}{\pi}\right)\left\{-\tilde{\beta}\left(\frac{k}{k^2+\beta^2}\right) + k\left(\frac{\beta}{k^2+\beta^2}\right)\right\} \\ &= 8a\tilde{a}_i\left(\frac{\beta}{2\pi}\right)^{1/2}\left[\frac{\beta-\tilde{\beta}}{k^2+\beta^2}\right]\end{aligned} \tag{55a}$$

and $$< \psi_{bound}(^3S)\left|\psi_{free}(^1S)\right. > \rightarrow \Im_e \tag{55b}$$

From Equ.(55) we can see that the reason why $< \psi_{bound}(^3S)\left|\psi_{free}(^1S)\right. >$ is not zero is that $\beta - \tilde{\beta}$ is non-zero, due to the different nuclear potentials in the two spin states. On the other hand, if the spin states had been the same, and hence $\beta = \tilde{\beta}$, Equ.(55) reproduces the required orthogonality demonstrated previously. Using Equ.(37b), the squared matrix element is,

$$\left|< \psi_{bound}(^3S)\left|\psi_{free}(^1S)\right. >\right|^2 = 4\left[\frac{\beta-\tilde{\beta}}{k^2+\beta^2}\right]^2 \frac{\beta}{\left(1+\tilde{B}/E\right)r_{max}} \tag{56}$$

2.4 The Free Nucleon-Nucleon P State

2.4.1 The (Free) P State Solution of the Schrodinger Equation

The solution is given by Equ.(57) and Equ.(58) below;

For $r \le a$, $\psi = a_i j_1(\rho_i) P_1^m e^{im\phi}$, where, $\rho_i = \alpha a$, $(\hbar\alpha)^2 = 2m(V_0 + E)$, $\xi = \alpha a$

For $r > a$, $\psi = \left[a_{es} j_1(\rho_e) + a_{ec} n_1(\rho_e)\right] P_1^m e^{im\phi}$, where, $\rho_e = ka$, $(\hbar k)^2 = 2mE$, $\kappa = ka$

where j_1 and n_1 are the two independent spherical Bessel functions of first order, i.e.,

$$j_1(\rho) = \frac{\sin\rho}{\rho^2} - \frac{\cos\rho}{\rho} \qquad n_1(\rho) = \frac{\cos\rho}{\rho^2} + \frac{\sin\rho}{\rho} \tag{59}$$

(the conventional definition of n_1 has the opposite sign to ours). In Equs.(57,58) we have used an un-normalised form of the spherical harmonics,

$$P_1^0 = \cos\theta, \qquad P_1^{+/-1} = \sin\theta \tag{60}$$

We shall be concerned only with the m = 0 solution, since the m = +/-1 solutions have zero electric dipole matrix element with the bound S-state by virtue of the azimuthal angle dependence, i.e. $<^3 \mathrm{S(bound)}|r\cos\theta|^3 \mathrm{P(free)}, m = +/-1 >= 0$.

The derivatives of the spherical Bessel functions may be written,

$$j_1'(\rho) = -2\frac{\sin\rho}{\rho^3} + 2\frac{\cos\rho}{\rho^2} + \frac{\sin\rho}{\rho}, \quad n_1'(\rho) = -2\frac{\cos\rho}{\rho^3} - 2\frac{\sin\rho}{\rho^2} + \frac{\cos\rho}{\rho} \tag{61}$$

The continuity conditions at r = a are,

$$a_i j_1(\xi) = a_{es} j_1(\kappa) + a_{ec} n_1(\kappa) \qquad a_i \alpha j_1'(\xi) = a_{es} k j_1'(\kappa) + a_{ec} k n_1'(\kappa) \tag{62}$$

2.4.2 Zero-Range Approximation For The Free ^{3}P State

To give a non-zero electric dipole matrix element for a transition between the ground triplet state of the deuteron and a free P-wave state requires the P-wave also to be a spin triplet, i.e. ^{3}P. [This is because $<^3 \mathrm{S(bound)}|r\cos\theta|^1 \mathrm{P(free)} >= 0$ because the spin states are orthogonal and the electric dipole interaction does not cause a spin flip]. Hence, in deploying the zero-range approximation it is the triplet binding energy, B, which is relevant. (This contrasts with Section 2.2.2 in which the free <u>singlet</u> S state was considered, and which therefore involves the singlet binding energy, B'). Thus, to recap, the zero-range limiting conditions are,

$$a \to 0, \quad \kappa = ka \to 0, \quad \xi \to \frac{\pi}{2} + \frac{2}{\pi}\beta a, \quad \alpha = \xi/a \to \frac{\pi}{2a} + \frac{2}{\pi}\beta \tag{63}$$

Hence, we find the limiting forms,

$$\frac{\sin\xi}{\xi} \to \frac{2}{\pi}\left(1 - \frac{4}{\pi^2}\beta a - \frac{2}{\pi^2}(\beta a)^2\right) + O(a^3) \tag{64a}$$

$$\frac{\sin\xi}{\xi^2} \to \left(\frac{2}{\pi}\right)^2\left(1 - \frac{8}{\pi^2}\beta a\right) + O(a^2) \tag{64b}$$

$$\frac{\sin\xi}{\xi^3} \to \left(\frac{2}{\pi}\right)^3\left(1 - \frac{12}{\pi^2}\beta a + \frac{2}{\pi^2}\zeta'(\beta a)^2\right) + O(a^3) \tag{64c}$$

$$\frac{\cos\xi}{\xi} \to -\left(\frac{2}{\pi}\right)^2 \beta a + O(a^2) \tag{64d}$$

$$\frac{\cos\xi}{\xi^2} \to -\left(\frac{2}{\pi}\right)^3 \beta a\left(1-\frac{8}{\pi^2}\beta a\right) + O(a^3) \tag{64e}$$

where $\zeta' = \left(48/\pi^2\right) - 1 \approx 3.8634$, so that,

$$j_1(\xi) \to \frac{4}{\pi^2}\left(1+\zeta\beta a\right) + O(a^2), \qquad \text{where } \zeta = 1-\frac{8}{\pi^2} \approx 0.1894 \tag{65}$$

$$j_1'(\xi) \to \frac{2}{\pi}\zeta\left(1-\frac{12}{\pi^2}\beta a - \frac{4\zeta''}{\pi^2}(\beta a)^2\right) + O(a^3) \tag{66}$$

where $\zeta\zeta'' = \frac{192}{\pi^4} - \frac{20}{\pi^2} + \frac{1}{2} = 0.4446$.

Similarly, taking the limit $\kappa = O(a) \to 0$ gives,

$$j_1(\kappa) \to \frac{\kappa}{3} + O(a^3) \qquad n_1(\kappa) \to \frac{1}{\kappa^2} + \frac{1}{2} + O(a^2) \tag{67}$$

$$j_1'(\kappa) \to \frac{1}{3} + O(a^2) \qquad n_1'(\kappa) \to -\frac{2}{\kappa^3} - \frac{\kappa}{4} + O(a^3) \tag{68}$$

Hence, the continuity equations (61,62) become, in the zero-range approximation,

$$\frac{4}{\pi^2}\left(1+\zeta\beta a\right)a_i = \frac{\kappa}{3}a_{es} + \left(\frac{1}{\kappa}+\frac{1}{2}\right)a_{ec} \tag{69a}$$

$$\frac{2}{\pi}\zeta\alpha\left(1-\frac{12}{\pi^2}\beta a - \frac{4}{\pi^2}\zeta''(\beta a)^2\right)a_i = \frac{k}{3}a_{es} - \left(\frac{2k}{\kappa^3}+\frac{\kappa}{4}\right)a_{ec} \tag{69b}$$

Solving Equs.(69a,b) gives,

$$a_{es} = \left[\frac{1}{\kappa} + \tilde{\zeta}\kappa + O(a^2)\right]a_i, \qquad a_{ec} = \left[\left(-\frac{1}{3}+\frac{4}{\pi^2}\right)\kappa^2 + O(a^3)\right]a_i \tag{70a}$$

where, $$\tilde{\zeta} = \left(\frac{1}{3}-\frac{4}{\pi^2}\right) - \left(\frac{2}{\pi}\right)^2 \zeta\left(\zeta'' + \frac{12}{\pi^2}\right)\left(\frac{\beta}{k}\right)^2 = -0.072 - 0.2735\left(\frac{\beta}{k}\right)^2 \tag{70b}$$

We can now determine the magnitude of the coefficients in the zero-range approximation by enforcing normalisation. In view of our earlier results we anticipate

that the interior integral will be negligible in this approximation. It will equal some numerical constant, A_i, which could be found by explicit integration, times $|a_i|^2 a^3$, i.e.,

$$\Im_i = A_i |a_i|^2 a^3 \tag{71}$$

The exterior integrals are,

$$\Im_{es} = |a_{es}|^2 \int_a^{r_{max}} (j_1(kr))^2 \cos^2\theta \cdot 2\pi d\theta \cdot r^2 dr = \frac{4\pi}{3} |a_{es}|^2 \int_a^{r_{max}} (j_1(kr))^2 r^2 dr \tag{72a}$$

$$\Im_{ec} = |a_{ec}|^2 \int_a^{r_{max}} (n_1(kr))^2 \cos^2\theta \cdot 2\pi d\theta \cdot r^2 dr = \frac{4\pi}{3} |a_{ec}|^2 \int_a^{r_{max}} (n_1(kr))^2 r^2 dr \tag{72b}$$

It is clear on dimensional grounds that, in the zero-range limit, we get,

$$\Im_{es} \to A_{es} |a_{es}|^2 k^{-2} r_{max} = \tilde{A}_{es} |a_i|^2 k^{-4} r_{max} / a^2 \tag{73a}$$

$$\Im_{ec} \to A_{ec} |a_{ec}|^2 k^{-2} r_{max} = \tilde{A}_{ec} |a_i|^2 r_{max} a^2 \tag{73b}$$

where the second form has used the zero-range limits for the coefficients, Equ.(70a). The sum of (71), (73a) and (73b) must be unity. Hence it is clear that a_i must be of order a as $a \to 0$, which makes the term $\Im_{es} \to 1$ and the terms $\Im_i, \Im_{ec} \to 0$. In fact the latter integrals are of order a^5 and a^4 respectively, and hence are negligible even as a first, second or third order correction. We now find the coefficient A_{es} by explicit integration of Equ.(72a). We have,

$$I_{es} = \int_a^{r_{max}} (j_1(kr))^2 r^2 dr = \frac{1}{k^3} \int_\kappa^{\rho_{max}} \left[\frac{\sin^2\rho}{\rho^2} - \frac{2\sin\rho\cos\rho}{\rho} + \cos^2\rho \right] d\rho \tag{74}$$

With the help of Gradshteyn 2.642(5), the first integral is found in the limit that $\kappa \to 0$ and $\rho_{max} \to \infty$ to be the same as,

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2} \tag{75}$$

By transforming to the integration variable $x = 2\rho$, the second integral in (74) is also found to equal (75), and hence cancels with the first. To evaluate the third integral we assume as we did in Section 2.2.1 that the range κ to ρ_{max} corresponds to a whole number of wavelengths. Hence, the third integral is simply $(\rho_{max}-\kappa)/2$. [Incidentally, this assumption also means that the density of states must be the same for the P-wave as for the S-wave]. Thus, we get $I_{es} = \rho_{max} / 2k^3$ to an accuracy of $O(\kappa)$ and hence,

$$\Im_{es} = \frac{2\pi\rho_{max}}{3k^3}\left|a_{es}\right|^2 = \frac{2\pi r_{max}}{3k^2}\left|a_{es}\right|^2 \tag{76}$$

and equating this to unity gives,

$$a_{es}^2 = \frac{3k^2}{2\pi r_{max}}, \quad a_i^2 = \frac{3a^2k^4}{2\pi r_{max}}, \quad a_{ec}^2 = \frac{3a^6k^8}{2\pi r_{max}} \tag{77}$$

[where a_i and a_{es} are positive, but a_{ec} is negative]. It is clear that we need be concerned only with the exterior j_1 (a_{es}) term from now on. We will see below that the second order correction is important as regards the coefficient of the *bound* state. However, the second order term is *not* important for the free state. Accounting for the finite size of κ, the first two integrals in (74) do not exactly cancel, but rather contribute a value $-\kappa + O(\kappa^3)$. Since this has been ignored we need to add κ to k^3I_{es} to correct it. However, the third integral also contributes an amount $-\kappa/2$. Hence, the overall correction to k^3I_{es} is to add $\kappa/2$ to it. Thus we get,

$$I_{es} = \frac{\rho_{max}}{2k^3} + \frac{\kappa}{2k^3} = \frac{\rho_{max}}{2k^3}\left(1 + \frac{\kappa}{\rho_{max}}\right) = \frac{\rho_{max}}{2k^3}\left(1 + \frac{a}{r_{max}}\right)$$

and because the correction term, linear in 'a', involves the ratio a/r_{max}, it is negligible.

2.4.3 Bound ^{3}S – Free ^{3}P Dipole Matrix Element

The above development of the free ^{3}P state has been motivated by the requirement to calculate the electric dipole matrix element,

$$<^3 S(bound)\left|\hat{H}'_{electric} \propto E_0 r\cos\theta\right|^3 P(free) > \tag{78}$$

We note in passing that the ^{3}S(bound) – ^{1}S(free) electric dipole matrix element is zero (the spin states are orthogonal, and also the angular integration in the spatial matrix element is zero). As for ^{3}S(bound) – ^{1}P(free) matrix element, the spatial part is non-zero, being identical to that for the ^{3}P state. However, the spin states are orthogonal – and, of course, the electric interaction Hamiltonian does not cause a spin flip. Hence, in this Section we calculate the matrix element (78) only.

It turns out that, unlike the ^{3}S-^{1}S magnetic dipole transition, it is important to work to second order in the zero-range approximation. We have seen, above, that the interior and exterior n_1 coefficients are negligible, hence we need consider only the exterior j_1 term. The matrix element, (78) is thus,

$$\begin{aligned} <^3 S\left|r\cos\theta\right|^3 P> &= \int_a^{r_{max}} a_e\frac{e^{-\beta r}}{\beta r}a_{es}\left(\frac{\sin kr}{(kr)^2} - \frac{\cos kr}{kr}\right)2\pi\cos^2\theta\cdot d(\cos\theta).r^2dr \\ &= \frac{4\pi a_e a_{es}}{3\beta k^2}\int_a^{r_{max}} e^{-\beta r}\left[\sin kr - kr\cos kr\right]dr = \frac{4\pi a_e a_{es}}{3\beta k^2}\left[\chi_s - \tilde{\chi}\right] \end{aligned} \tag{79}$$

where integration by parts gives, $\tilde{\chi} = \dfrac{ak(\beta\cos ka - k\sin ka)e^{-\beta a} - k^2\chi_s + \beta k\chi_c}{k^2+\beta^2}$ and the integrals χ_s and χ_c are given by Equs.(42,43). The zero-range approximation for these integrals, retaining terms linear in 'a', are,

$$\chi_s \to \frac{k}{k^2+\beta^2} + O(a^2); \quad \chi_c \to \frac{\beta}{k^2+\beta^2} - a + O(a^2); \tag{80}$$

$$\tilde{\chi} \to \frac{k\left(\beta^2 - k^2\right)}{\left(\beta^2+k^2\right)^2} + O(a^2) \tag{81}$$

The important thing is that the correction terms linear in 'a' cancel from both χ_s and $\tilde{\chi}$, and hence there is no linear correction term to the matrix element (79) from the integral itself. We have also seen from Section 2.4.2 that there is no linear 'a' correction term in the free state coefficient, a_{es}. Hence, the linear 'a' correction arises entirely from the correction to the bound state a_e coefficient, as given by Equ.(21b).

Substituting Equs.(80,81) into (79) gives,

$$<^3 S\left|r\cos\theta\right|^3 P>= \frac{8\pi a_e a_{es} k}{3\beta\left(k^2+\beta^2\right)^2} \tag{82}$$

and substituting for the coefficients from Equ.(21b) and Equ.(77) gives,

$$\begin{aligned}\left|<^3 S\left|r\cos\theta\right|^3 P>\right|^2 &= \left[\frac{8\pi k}{3\beta\left(k^2+\beta^2\right)^2}\right]^2 \left(\frac{3k^2}{2\pi r_{max}}\right)\left(\frac{\beta^3}{2\pi}\right)\left(1+\frac{8}{\pi^2}\beta a\right) \\ &= \frac{16\beta k^4}{3\left(k^2+\beta^2\right)^4 r_{max}}\left(1+\frac{8}{\pi^2}\beta a\right)\end{aligned} \tag{83}$$

noting that the matrix element has dimensions of length.

3. The Cross Sections In Terms Of The Matrix Elements

Schiff describes the derivation of Fermi's "Golden Rule #2". We envisage a bound state being exposed to electromagnetic radiation so as to induce photodisintegration. Schiff's Equ.(35.14) gives the probability of a given composite object (e.g. a deuteron) disintegrating per second (w) in terms of the density of final states per unit energy range ($d\rho/dE$) and the relevant matrix element, i.e.,

$$w = \frac{2\pi}{\hbar}\cdot\frac{d\rho}{dE}\left|<\psi_{in}\left|H'\right|\psi_{out}>\right|^2 \tag{100}$$

where H' is a time independent interaction Hamiltonian (energy). [NB: Schiff's ρ is our $d\rho/dE$]. Schiff's derivation of Equ.(100) requires that the time independent interaction Hamiltonian is related to the actual interaction Hamiltonian by,

$$< \psi_{in} \left| H'(t) \right| \psi_{out} >= 2 < \psi_{in} \left| H' \right| \psi_{out} > \sin \omega t \qquad (101)$$

where ω is the (angular) frequency of the radiation. The salient feature of Equ.(101) is the factor of 2.

3.1 Density of States

In this Section we derive expressions for the density of final states. Recall that we are envisaging the photodisintegration of a bound state into a pair of nucleons, so the final state is that of a pair of free nucleons.

Density of S States

Recall that, in order to simplify the algebra resulting from normalisation, we have already effectively assumed a discrete set of free states – see Equ.(28). These states are equally spaced along the k-axis, with a spacing of $\Delta k = \pi/(r_{max} - a) \to \pi / r_{max}$, the latter applying in the zero-range approximation. Thus the number of states per unit k range is one per Δk. Since the (total) energy of the pair of nucleons is given by $E = (\hbar k)^2 / 2m$, where m is the reduced mass, we have $dE = \hbar^2 k dk / m$. Thus, the density of states per unit energy interval is,

$$\frac{d\rho}{dE} = \frac{d\rho}{dk} \cdot \frac{dk}{dE} = \frac{1}{\Delta k} \cdot \frac{m}{\hbar^2 k} = \frac{r_{max}}{\pi} \cdot \frac{m}{\hbar^2 k} \qquad (102)$$

Density of P States

This needs checking, but I suspect it will be the same as Equ.(102).

3.2 Photodisintegration Interaction Hamiltonians

We shall work only within the approximation that the wavelength of the electromagnetic radiation is long compared with the size of the region within which there is a significant change of finding the nucleons in the bound state. Recalling that 'a' is roughly 2fm, and that the decay length in the exterior region is $1/\beta = \hbar / \sqrt{2mB}$ = 4.3fm for the deuteron, we require radiation with wavelength greater than about 10fm, i.e. photon energies less than about 20MeV. Since it requires a photon energy of 2.224MeV to disintegrate a deuteron, we are implicitly considering the specific photon energy range 2.3MeV – 20MeV, or so.

Thus, we can consider the electric and magnetic fields comprising the electromagnetic radiation to be uniform over the size scale of interest. The electric component of the interaction energy is thus,

$$H'_{electric} = N_p e E_0 z = N_p e E_0 r \cos\theta \qquad (104)$$

where, in line with Equ.(101), H' and hence E_0 are the amplitudes of the radiation wave. We imagine (104) to be the potential energy loss, or gain, if the nucleons undergo a relative displacement by 'z' in the uniform electric field E_0. The polar z direction is therefore the polarisation (E-vector) of the radiation. The charge 'e' is taken as positive since we are dealing with a proton. Finally, N_p is the number of

protons. Thus, $N_p = 1$ for the deuteron, $N_p = 2$ for the diproton, and $N_p = 0$ for the dineutron. Thus, there is no electric dipole interaction for the dineutron, but the electric dipole interaction Hamiltonian is twice as great for the diproton as for the deuteron.

The magnetic field couples to the magnetic dipole moments caused by the nucleon spins. Denoting a Pauli-type operator by $\hat{\sigma}$, representing the spin in some fixed Cartesian direction (but with eigenvalues +1 and –1) we may write the magnetic moment operator for a two nucleon system as,

$$\hat{\mathrm{M}}\mathrm{ag} = \left(\mu_1\hat{\sigma}_1 + \mu_2\hat{\sigma}_2\right)\frac{e\hbar}{2M_p} \tag{105}$$

where $\mu_{1,2}$ are 2.7934 and –1.9135 for the proton and neutron (but note that particles 1 and 2 may both be neutrons or both protons, or one of each). Also note that the gyromagnetic ratios are <u>twice</u> the $\mu_{1,2}$, because the nucleon spin is $\hbar/2$. The magnetic dipole interaction Hamiltonian operator may thus be written,

$$H'_{\mathrm{magnetic}} = \hat{\mathrm{M}}\mathrm{ag}\cdot B_0 \tag{106}$$

where the direction of the magnetic moment operator is chosen to align with the applied magnetic field (or, generally, the dot product is taken). Note that (106) again involves the amplitude of the magnetic field, in line with (101). Thus, we see from Equ.(100) that the magnetic dipole cross-section depends on the matrix element,

$$<\psi_{\mathrm{in}}\left|H'_{\mathrm{magnetic}}\right|\psi_{\mathrm{out}}> = <\psi_{\mathrm{in}}^{\mathrm{spatial}}\left|\psi_{\mathrm{out}}^{\mathrm{spatial}}\right.><\psi_{\mathrm{in}}^{\mathrm{spin}}\left|H'_{\mathrm{magnetic}}\right|\psi_{\mathrm{out}}^{\mathrm{spin}}> \tag{107}$$

where we have written the wavefunctions as a product of a spatially varying part and the spin part so as to illustrate that the magnetic dipole interaction involves only the spin part [because (106) is uniform in space – in the approximation that the photon wavelength is large compared with the size of the region within which the bound nucleons might be found]. The spatial matrix element between S states has been evaluated in Section 2. We now evaluate the spin matrix element.

We denote by $|1_\uparrow 2_\downarrow>$, say, a state in which particle 1 has spin up and particle 2 has spin down, etc. By virtue of (106) we shall take the spin z direction to be the local magnetic field direction. So, for example, we would have,

$$\hat{\mathrm{M}}\mathrm{ag}|1_\uparrow 2_\downarrow> = \left(\mu_1\hat{\sigma}_1 + \mu_2\hat{\sigma}_2\right)\frac{e\hbar}{2M_p}|1_\uparrow 2_\downarrow> = \left(\mu_1 - \mu_2\right)\frac{e\hbar}{2M_p}|1_\uparrow 2_\downarrow> \tag{108}$$

The states which are of interest are the eigenstates of the total spin operator. These are (see any standard text),

$$|\mathrm{S}> = \frac{1}{\sqrt{2}}\left[|1_\uparrow 2_\downarrow> - |1_\downarrow 2_\uparrow>\right] \qquad |\mathrm{T},0> = \frac{1}{\sqrt{2}}\left[|1_\uparrow 2_\downarrow> + |1_\downarrow 2_\uparrow>\right] \tag{109a}$$

$$\left|\mathrm{T},+1\right> = \left|1_{\uparrow}2_{\uparrow}\right> \qquad\qquad \left|\mathrm{T},-1\right> = \left|1_{\downarrow}2_{\downarrow}\right> \tag{109b}$$

where 'S' and 'T' stand for singlet and triplet respectively. We can now evaluate the spin matrix element in (107). We note firstly that we are only interested in the matrix elements for which the in and out states differ, i.e. one is singlet and the other triplet. The spin matrix element could be non-zero even for identical in and out spin states (specifically if the states are T,+/-1, but are zero for T,0 and S). However, the spatial matrix element in (107) would then be zero by orthogonality, as we have seen in Section 2. That is,

$$\psi_{\mathrm{in}}^{\mathrm{spin}} = \psi_{\mathrm{out}}^{\mathrm{spin}} \Rightarrow \left<\psi_{\mathrm{in}}^{\mathrm{spatial}}\middle|\psi_{\mathrm{out}}^{\mathrm{spatial}}\right> = 0 \tag{110}$$

We firstly note that the spin matrix element between a singlet state and a triplet state with non-zero azimuthal quantum number ($m \neq 0$) is zero, i.e.,

(111)

$$\begin{aligned}
\left<\psi_{\mathrm{in}}^{\mathrm{spin}}\middle|H'_{\mathrm{magnetic}}\middle|\psi_{\mathrm{out}}^{\mathrm{spin}}\right> &= \left<\mathrm{T},+1\middle|\left(\mu_1\hat{\sigma}_1+\mu_2\hat{\sigma}_2\right)\frac{e\hbar B_0}{2M_p}\middle|\mathrm{S}\right> \\
&= \frac{1}{\sqrt{2}}\left<1_{\uparrow}2_{\uparrow}\middle|\left(\mu_1\hat{\sigma}_1+\mu_2\hat{\sigma}_2\right)\frac{e\hbar B_0}{2M_p}\middle|\left[\left|1_{\uparrow}2_{\downarrow}\right>-\left|1_{\downarrow}2_{\uparrow}\right>\right]\right. \\
&= \frac{1}{\sqrt{2}}\frac{e\hbar B_0}{2M_p}\left\{\left<1_{\uparrow}2_{\uparrow}\middle|\left(\mu_1-\mu_2\right)\middle|1_{\uparrow}2_{\downarrow}\right>\middle|-\left<1_{\uparrow}2_{\uparrow}\middle|\left(-\mu_1+\mu_2\right)\middle|1_{\downarrow}2_{\uparrow}\right>\middle|\right\} \\
&= 0
\end{aligned}$$

We are therefore left with the <singlet – triplet,m=0> matrix element as being the only one of interest. This is,

(112)

$$\begin{aligned}
\left<\psi_{\mathrm{in}}^{\mathrm{spin}}\middle|H'_{\mathrm{magnetic}}\middle|\psi_{\mathrm{out}}^{\mathrm{spin}}\right> &= \left<\mathrm{T},0\middle|\left(\mu_1\hat{\sigma}_1+\mu_2\hat{\sigma}_2\right)\frac{e\hbar B_0}{2M_p}\middle|\mathrm{S}\right> \\
&= \left(\frac{1}{\sqrt{2}}\right)^2\left[\left<1_{\uparrow}2_{\downarrow}\right|+\left<1_{\downarrow}2_{\uparrow}\right|\right]\left(\mu_1\hat{\sigma}_1+\mu_2\hat{\sigma}_2\right)\frac{e\hbar B_0}{2M_p}\left|\left[\left|1_{\uparrow}2_{\downarrow}\right>-\left|1_{\downarrow}2_{\uparrow}\right>\right]\right. \\
&= \frac{e\hbar B_0}{4M_p}\left\{\left<1_{\uparrow}2_{\downarrow}\middle|\left(\mu_1-\mu_2\right)\middle|1_{\uparrow}2_{\downarrow}\right>\middle|-\left<1_{\downarrow}2_{\uparrow}\middle|\left(-\mu_1+\mu_2\right)\middle|1_{\downarrow}2_{\uparrow}\right>\middle|\right\} \\
&= \frac{e\hbar B_0}{2M_p}\left(\mu_1-\mu_2\right)
\end{aligned}$$

Thus we see that the spin matrix element of interest is proportional to the *difference* in the two particles' magnetic moments, $\mu_1 - \mu_2$. It follows that there will be no magnetic dipole transition for identical particles, or any particles with the same magnetic moment. Note, however, that we are working to first order in the perturbation, H', and have also ignored the spatial variation of the electromagnetic

field. Relaxing one or both of these approximations may result in a magnetic transition even between identical particles. This is beyond our scope.

3.3 Electromagnetic Units And Photon Density

Conversion of the transition probability rate of Equ.(100) to a cross-section requires the flux of photons corresponding to the interaction Hamiltonian H' to be evaluated. Before doing this we recap some basic electromagnetic theory, together with the MKSA units. In the following, [x] denotes "the units or dimension of x". The notation for units and dimensions will be,

M = mass; L = Length; T = time; C = Charge
kg = kilogram; m = metres; s = seconds C = Coulomb

The ambiguity in 'C' will not matter since they relate to the same quantity. Note, however, the important distinction between M and m.

N = Newton = $kg.ms^{-2}$; J = Nm
Fa (Farad) = C^2J^{-1} He (Henry) = $JC^{-2}s^2 = s^2/Fa$

E = Electric field; Force = qE implies $[E] = NC^{-1}$

$$E = \frac{q}{4\pi\varepsilon_0 r^2} \text{ implies } [\varepsilon_0] = C^2J^{-1}m^{-1} = Fa/m: \qquad 4\pi\varepsilon_0 = 10^7/c^2 \text{ Fa/m}$$

B = Magnetic field (flux density); Force = Bqv implies $[B] = NC^{-1}m^{-1}s = kgC^{-1}s^{-1}$

$$\delta B = \frac{\mu_0}{4\pi}\cdot\frac{I\delta\bar{L}\times\hat{r}}{r^2} \text{ implies } [\mu_0] = NC^{-2}s^2 = He/m: \qquad \frac{\mu_0}{4\pi} = 10^{-7} \text{ He/m}$$

$$\mu_0\varepsilon_0 = 1/c^2 \ (s/m)^2$$

Maxwell's Equations (vacuum):-

$$(1)\ c\bar{\nabla}\cdot\bar{B} = 0 \qquad (2)\ \bar{\nabla}\cdot\bar{E} = \rho/\varepsilon_0 \tag{113a}$$

$$(3)\ \bar{\nabla}\times\bar{E} + \frac{\partial\bar{B}}{\partial t} = 0 \qquad (4)\ c\bar{\nabla}\times\bar{B} - \frac{1}{c}\frac{\partial\bar{E}}{\partial t} = \frac{\bar{J}}{c\varepsilon_0} \tag{113b}$$

where ρ and J are charge density and current flux. It is simply checked that all terms in the four equations, written as they are above, have units $Nm^{-1}C^{-1}$.

In a gauge in which the scalar potential is zero, $\bar{B} = \bar{\nabla}\times\bar{A}$ and $\bar{E} = -\partial\bar{A}/\partial t$. Maxwell's equations (1) and (3) are then identities. A convenient representation of a plane electromagnetic wave is,

$$\bar{A} = \bar{A}_0\left[e^{i(\bar{k}\cdot\bar{r}-\omega t)} + e^{-i(\bar{k}\cdot\bar{r}-\omega t)}\right] = 2\bar{A}_0\cos(\bar{k}\cdot\bar{r}-\omega t) \quad \text{with} \quad \bar{A}_0\cdot\bar{k} = 0 \quad \text{and} \quad ck = \omega \tag{114}$$

which ensures all the Maxwell equations are obeyed, and for which,

$$\bar{E} = -2\omega\bar{A}_0 \sin\left(\bar{k}\cdot\bar{r} - \omega t\right) \quad \text{and} \quad \bar{B} = -2\bar{k}\times\bar{A}_0 \sin\left(\bar{k}\cdot\bar{r} - \omega t\right) \tag{115}$$

At r = 0 (115) reduces to (101). Note that $\bar{k}$ is in the direction of wave propagation, $\bar{A}_0$ is in the direction of the polarisation (electric field vector), and $\bar{B}$ is perpendicular to both. Note also that the matrix elements of H' which enter the cross-section formula are defined by amplitudes E_0 and B_0 which are only half the true amplitude of the electromagnetic wave, i.e.,

$$E_0 = \omega A_0 \text{ and } B_0 = kA_0 \qquad \text{NB: } [A] = NC^{-1}s. \tag{116}$$

$$\text{Electromagnetic energy density, } \xi = \frac{\varepsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2 \ , \ \left[\xi\right] = Jm^{-3} \tag{117}$$

$$\text{Electromagnetic energy flux, } \bar{J}_{em} = \frac{1}{\mu_0}\bar{E}\times\bar{B}, \ \left[J_{em}\right] = Jm^{-2}s^{-1} \tag{118}$$

We are now in a position to evaluate the photon flux (in photons per square meter per second) corresponding to the plane wave of Equs.(115). From (118) the energy flux (Joules per square metre per second) at the origin (r=0) is,

$$J_{em} = \frac{4\omega k A_0^2}{\mu_0}\sin^2\omega t = \frac{4E_0^2}{c\mu_0}\sin^2\omega t = \frac{4cB_0^2}{\mu_0}\sin^2\omega t \tag{119}$$

The time average value is thus,

$$< J_{em} >= \frac{2E_0^2}{c\mu_0} = \frac{2cB_0^2}{\mu_0} \tag{120}$$

Since the energy per photon is $\hbar\omega$ the photon flux corresponding to a monochromatic, unidirectional plane wave is therefore simply,

$$< J_\gamma^N >= \frac{2E_0^2}{\hbar\omega c\mu_0} = \frac{2cB_0^2}{\hbar\omega\mu_0} \tag{121}$$

where the superscript 'N' denotes a number density, i.e. photons per square metre per second.

Finally, the dimensionless electromagnetic fine structure constant is given by,

$$\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} = \frac{\mu_0 c e^2}{4\pi\hbar} = \frac{1}{137} \tag{122}$$

Finally, we also note that the dimensions of a magnetic dipole moment, defined as the product of current and area, are [m] = Cm^2s^{-1}, and this agrees with the dimensions of

the factor $\hbar e / 2M_p$ used to non-dimensionalise the magnetic moments of the nucleons. This also confirms that [mB] = energy.

4. Evaluation Of The Cross-Sections

4.1 Deuteron Photodisintegration: Magnetic Dipole Cross-Section

We have seen in Section 3.2 that the magnetic dipole matrix element between the bound ^{3}S state and a free ^{1}S state is non-zero only if the azimuthal quantum number (m) is zero, i.e. only if the deuteron has zero spin component parallel to the magnetic field, B. We shall assume that we are dealing with unpolarised deuterons. In this case only one in three deuterons will be favourably aligned with the incident radiation's B-direction (because the three triplet states are equally likely). The formula for the unpolarised disintegration cross-section is therefore derived from the transition probability per second, Equ.(100), by dividing by the photon flux, Equ.(121), and also by a factor of 3. Explicitly including the density of states from Equ.(102), gives,

$$\sigma_M^{dis} = \frac{1}{3}\frac{2\pi}{\hbar}\cdot\frac{mr_{max}}{\pi\hbar^2 k}\cdot\frac{\hbar\omega\mu_0}{2cB_0^2}\left|<\psi_{in}\left|H'\right|\psi_{out}>\right|^2 \tag{123}$$

where k relates to the nucleons, i.e. $W = (\hbar k)^2 / 2m$ is the total nucleon energy, whereas ω relates to the photon, i.e. the photon energy is $\hbar\omega = W + B$. Recall that m is the reduced mass (roughly $M_p/2$). The squared matrix element is given by the product of the spin part, Equ.(112), and the spatial part, Equ.(56), i.e.,

$$\left|<\psi_{in}\left|H'\right|\psi_{out}>\right|^2 = \left[\frac{e\hbar B_0}{2M_p}(\mu_1-\mu_2)\right]^2 4\left[\frac{\beta-\tilde{\beta}}{k^2+\beta^2}\right]^2\frac{\beta}{(1+\tilde{B}/E)r_{max}} \tag{124}$$

Substituting (124) into (123), and noting that the two particles are protons and neutrons in this case, gives,

$$\sigma_M^{dis} = \frac{1}{3}\frac{2\pi}{\hbar}\cdot\frac{mr_{max}}{\pi\hbar^2 k}\cdot\frac{\hbar\omega\mu_0}{2cB_0^2}\left[\frac{e\hbar B_0}{2M_p}(\mu_p-\mu_n)\right]^2 4\left[\frac{\beta-\tilde{\beta}}{k^2+\beta^2}\right]^2\frac{\beta}{(1+\tilde{B}/E)r_{max}} \tag{125}$$

where μ_p = 2.7928, μ_n = -1.9130, so that μ_p - μ_n = 4.7058. Also recall from equ.(34c) and the associated discussion that $2m\tilde{B} = (\hbar/{}^1a)^2$ and $\tilde{\beta} = 1/{}^1a$ where ^{1}a is the singlet scattering length of the deuteron (-23.69fm). Also, following Evans, we define the "radius of the deuteron" as $\rho = 1/\beta = 4.32$fm. Finally, for convenience we define a length parameter by,

$$L = \frac{\hbar(\mu_p-\mu_n)}{2mc} \tag{126}$$

noting that it is the reduced mass which appears in Equ.(126). Using (122) for the fine structure constant, Equ.(125) reduces to,

$$\sigma_M^{dis} = \frac{2\pi}{3}\alpha L^2\left(\frac{\sqrt{WB}}{W+B}\right)\eta_M, \quad \text{where} \quad \eta_M = \frac{\left(1-{}^1a/\rho\right)^2}{1+\left({}^1ak\right)^2} \tag{127}$$

It is important to recall that the singlet scattering length is negative (i.e. unbound), which results in the numerator of η_M being ~42. If the singlet state were bound (and hence ^{1}a positive), the disintegration cross section for the bound triplet state would thus be smaller.

4.2 Deuteron Photodisintegration: Electric Dipole Cross-Section

In the same manner as in Section 4.2, the electric dipole transition cross-section may be written, using Equs.(100, 102, 121),

$$\sigma_E^{dis} = \frac{2\pi}{\hbar}\cdot\frac{mr_{max}}{\pi\hbar^2 k}\cdot\frac{\hbar\omega c\mu_0}{2E_0^2}\left|<\psi_{in}\left|H'\right|\psi_{out}>\right|^2 \tag{128}$$

There is no polarisation factor of 1/3 in this case, since the spin direction of the deuteron is irrelevant for the electric dipole transition. The direction of the final orbital (L) state of the free neutron and proton must align with the spin direction (polarisation) of the initial photon. This means that only one of the L = 1 orbital states (namely m =0) is possible. But that has been accounted for in using the same density of states expression as for the S-waves, Equ.(102), since this implicitly assumes just one spin state (i.e. it gives the density of spatial states only).

Substituting into (128) for the matrix element from Equ.(83) [times $(eE_0)^2$] and also using Equ.(122) for the fine structure constant, gives,

$$\begin{aligned}\sigma_E^{dis} &= \frac{2\pi}{\hbar}\cdot\frac{mr_{max}}{\pi\hbar^2 k}\cdot\frac{\hbar\omega c\mu_0}{2E_0^2}\cdot e^2E_0^2\cdot\frac{16}{3}\frac{\beta k^4}{\left(k^2+\beta^2\right)^4 r_{max}}\left(1+\frac{8}{\pi^2}\beta a\right)\\ &= \frac{16}{3}\cdot 4\pi\alpha\frac{m(W+B)}{\hbar^2}\cdot\frac{\beta k^3}{\left(k^2+\beta^2\right)^4}\left(1+\frac{8}{\pi^2}\beta a\right)\\ &= \frac{32\pi}{3}\alpha\rho_D^2\left(\frac{\sqrt{BW}}{B+W}\right)^3\left(1+\frac{8}{\pi^2}\frac{a}{\rho_D}\right)\end{aligned} \tag{129}$$

where the "radius of the deuteron" is $\rho_D = 1/\beta = \hbar/\sqrt{2mB}$, recalling that m is the reduced mass, i.e. $m \approx (M_p + M_n)/4$. We have replaced the incoming photon energy $\hbar\omega$ with B+W in the above, recalling that W is the sum of the kinetic energies of the two free nucleons (in the CoM system).

Equ.(129) appears to differ from Evans' Chapter 10, Equ.(4.21) by a factor of 4. Why? There must be something amiss in my derivation. Also note that the correction term in (129) contains a factor of $8/\pi^2$ which Evans' expression does not have. This might be because the range of the square-well potential is not quite the same as the "effective range" used by Evans. This should be demonstrable using Evans' Equ.(3.19).

4.3 np Capture Cross Sections

The (corrected) photodisintegration cross sections of the deuteron are, following Evans,

$\underline{D + \gamma \rightarrow p + n}$:-

$$\sigma_{dis}^{M} = \frac{2\pi}{3}\alpha L_{M}^{2}\eta_{M}\frac{\sqrt{BW}}{B+W} \tag{130}$$

$$\sigma_{dis}^{E} = \frac{8\pi}{3}\alpha\rho_{3}^{2}\eta_{E}\left(\frac{\sqrt{BW}}{B+W}\right)^{3} \tag{131}$$

Where Evans expresses the $p + n \Leftrightarrow D + \gamma$ cross-sections in terms of the following,

M = the reduced mass;

B_3 = the binding energy of the deuteron (triplet state);

W = the total kinetic energies of the neutron and proton in the centre of mass system (this applies whether the nucleons are the incident particles or the products). The photon energy is given by $hf \cong W + B$ for energies small compared with M_n;

k = the relative wavenumber of the nucleons, such that $\hbar^2 k^2 \approx 2MW$ for energies small compared with M_n;

ρ_3 = the "radius of the deuteron" = $\hbar / \sqrt{2MB_3}$

r_{01} = effective range of the singlet nuclear force = 2.7 fm

r_{03} = effective range of the triplet nuclear force = 1.7 fm

The triplet scattering length a_3 is given by,

$$\frac{1}{a_3} = \frac{1}{\rho_3} - \frac{r_{03}}{2\rho_3^2}$$

In this universe the singlet spin state of the deuteron is not bound. This is reflected in the negative sign of the singlet scattering length, a_1 = -23.69 fm. However, for the purposes of our investigations in this Section we shall assume a relation like (17) applies also for the singlet state if this is bound, i.e.,

$$\frac{1}{a_1} = \frac{1}{\rho_1} - \frac{r_{01}}{2\rho_1^2} \qquad \text{where, } \rho_1 = \hbar / \sqrt{2MB_1}$$

and, in obvious notation, B_1 is the binding energy of the singlet state (this applying only for a nuclear force which has been increased sufficiently to bind the singlet deuteron state).

To derive the np capture cross-sections we use the general relation between a reaction and its inverse, i.e.,

$$\frac{\sigma(a+A\rightarrow b+B)}{\sigma(b+B\rightarrow a+A)}=\left(\frac{p_b}{p_a}\right)^2\frac{\text{number of spin states of } b+B}{\text{number of spin states of } a+A} \qquad (132)$$

where p_a is the 3-momentum of particle 'a' in the centre-of-mass (CoM) system, so that $p_a = p_A$, and similarly $p_b = p_B$ is the 3-momentum of particle 'b' (or 'B') in the CoM system. We note that the RHS of (135) is just the ratio of the density of states available to b + B as compared with a + A. Thus, Equ.(135) expresses the phase-space advantage, or disadvantage, that one configuration has over the other. This phase space (dis)advantage is the sole difference between the forward and backward cross-sections (i.e. the dynamical part of the cross section, as determined from the Feynman diagram, is the same for both). In the case of interest Equ.(135) becomes,

$$\frac{\sigma(n+p\rightarrow D+\gamma)}{\sigma(D+\gamma\rightarrow n+p)}=\left(\frac{p_\gamma}{p_n}\right)^2\frac{3x2}{2x2}=\frac{3}{2}\left(\frac{p_\gamma}{p_n}\right)^2 \qquad (133)$$

The exact relativistic formulae relating the required 3-momenta to the quantities W and B in terms of which the above cross-sections have been expressed are,

$$p_n^2=2MW+\frac{W^2}{4c^2} \qquad \text{and} \qquad p_\gamma c+\sqrt{p_\gamma^2c^2+M_D^2c^4}=W+(M_n+M_p)c^2 \qquad (134)$$

where the subscripts denote the deuteron, neutron and proton masses, so that the deuteron binding energy is just $B=\left[-M_D+(M_n+M_p)\right]c^2$. Recall that M is the reduced mass, and so is roughly half M_p. The above formulae from Evans have been derived assuming kinetic energies which are sufficiently small compared with the nucleon mass ($M_p \sim 938.27$MeV). [NB: At 1 second, $3kT \sim$ 2.6 MeV < 0.3%M_p, so this approximation is fine for our purposes]. In this case, Equs.(137) simplify to,

$$p_n^2\approx 2MW \qquad \text{and} \qquad p_\gamma c=W+B \qquad (135)$$

as assumed by Evans. Using (138), Equ.(136) becomes,

$$\frac{\sigma(n+p\rightarrow D+\gamma)}{\sigma(D+\gamma\rightarrow n+p)}=\frac{3}{2}\cdot\frac{(W+B)^2}{2Mc^2W} \qquad (136)$$

Hence, using Equ.(136) with Equ.(130) or (131) gives the capture cross-sections as,

$\underline{p+n\rightarrow D+\gamma}$:-

$$\sigma_{cap}^{M}=\pi\alpha L_M^2\eta_M\frac{\sqrt{B}(W+B)}{2Mc^2\sqrt{W}} \qquad (137)$$

where, $$L_M=\frac{\hbar(\mu_p-\mu_n)}{2Mc} \qquad (137a)$$

and, $$\eta_M = \frac{(1 - a_1/\rho_3)^2}{1 + (a_1 k)^2} \quad (137b)$$

and where the gyromagnetic moments are given by $\mu_p = 2.7934$ and $\mu_n = -1.9135$, and α is the electromagnetic fine-structure constant ($e^2/\hbar c$) as usual. Note that the dimensions of the cross-section come from the LM term, which has dimensions of length.

$$\sigma_{cap}^{E} = 4\pi\alpha\rho_3^2\eta_E\left[\frac{B\sqrt{BW}}{2Mc^2(B+W)}\right] \quad (138)$$

where, $$\eta_E = \frac{1}{1 - r_{03}/\rho_3} \quad (138a)$$

Note that the dimensions of the 'E' cross-section come from the "radius of the deuteron" term ρ_3.

These capture cross-sections may be checked against the capture reaction rate given by Hoffman et al. They give the reaction rate per neutron per second for an assumed proton density of one mole per cm^3. [NB: This is, of course, the same as the reaction rate per proton per second for an assumed neutron density of one mole per cm^3]. Rather surprisingly at first sight, this reaction rate is the same at all temperatures between 10^6 °K and 10^9 °K. Actually this follows from our cross-section, Equ.(137) as follows:-

For the temperature range of interest, $a_1 k$ may be neglected compared with unity (strictly it is of the order of unity at 10^9 °K, but is negligible at 10^8 °K and below). Thus, the term η_M reduces to the constant 42.2. Moreover, in this temperature range the maximum thermal energies (1.5kT) are of order 0.13 MeV, and hence small compared with the deuteron binding energy (2.22 MeV). Hence we replace B + W in the numerator of Equ.(19) with B. Finally, since we are dealing with non-relativistic speeds, the centre-of-mass total energy (W) is $M_n v^2/4 = Mv^2/2$, where v is the relative velocity of the incident proton and neutron. [NB: W is half the kinetic energy in the 'lab' frame, in which one of the nucleons is at rest]. Hence, with these approximations, Equ.(137) becomes,

$$\sigma_{cap}^{M} = 21.1\sqrt{2}\,\pi\alpha L_M^2\left(\frac{B}{Mc^2}\right)^{3/2}\frac{c}{v} = \frac{93.8}{137}\cdot(0.991\text{fm})^2\left(\frac{2.22}{469.46}\right)^{3/2}\frac{c}{v} = 2.2\frac{c}{v}\ (\mu\text{b}) \quad (139)$$

where μb = microbarns, and one barn = 10^{-28} m^2 = 100 fm^2. Thus, (139) gives,

$$v\sigma_{cap}^{M} = 6.60 \times 10^{-26}\ \ m^3/\text{sec} \quad (140)$$

Now the reaction rate per neutron per second is given by $\rho_p v\sigma_{cap}$ where ρ_p is the proton number density. Thus, for an assumed proton density of one mole per cm^3 (i.e. $\rho_p = 6.03 \times 10^{23} / (10^{-2}\ m)^3 = 6.03 \times 10^{29}\ m^{-3}$), Equ.(140) gives the reaction rate to be

$6.60 \times 10^{-26} \times 6.03 \times 10^{29} = 4.0 \times 10^{4}$ per second. This compares well with the reaction rate given by Hoffman et al of 4.37×10^{4} per second (per mole/cm^3), confirming that our derivation appears to be reliable.

Strictly, we have not yet checked that the 'E' capture cross-section is negligible in comparison. For the approximations introduced above, Equ.(138) reduces to,

$$\sigma_{cap}^{E} = \frac{20.8}{137}(4.31\text{fm})^2 \times 0.024\frac{v}{c} = 0.068\frac{v}{c}\ (\text{fm})^2 = 680\frac{v}{c}\ (\mu\text{b}) \qquad (141)$$

Unlike the 'M' cross-section, (139), the 'E' cross-section increases with energy (or speed, v). Hence it produces an energy dependent reaction rate. For the assumed one mole per cm^3 particle density the reaction rate due to the 'E' mechanism is given for a range of temperatures below,

T (^{o}K)	**v (m/s)**	**v/c**	**σ_{cap}^{E} (μb)**	**Reaction rate due to 'E' mechanism only (/sec)**
10^9	5×10^6	0.0166	11	3,300
10^8	1.6×10^6	0.0052	3.5	330
10^7	5×10^5	0.00166	1.1	33
10^6	1.6×10^5	0.00052	0.35	3.3

These compare with the 'M' mechanism reaction rate of 40,000 /sec, derived above, and hence are confirmed to be negligible at all but the highest energy considered where the contribution of the 'E' mechanism is ~8%. At this highest energy the total ('E' + 'M') reaction rate agrees with Hoffman et al almost precisely (43,300 /sec).

A further check on the capture cross-sections may be carried out at very low energies, corresponding to room temperature, i.e. 0.025 eV, for which $v/c = 7.3 \times 10^{-6}$. Equ.(139) then gives the 'M' capture cross-section as 0.30 barns, in agreement with Evans. In stark contrast, the 'E' mechanism cross-section from Equ.(141) is only 5×10^{-9} barns, again in agreement with Evans (P.338, 'Problem' answer).

Finally, we check the photodisintegration cross-sections against the results plotted in Evans Chapter 10, Figure 4.1. The results are obtained in straight forward fashion from Equs.(130, 131). We must only notice that the k-dependence in the η_M term in Equ.(130) leads to the 'M' capture cross section rising to a maximum very steeply at modest nucleon energies. We find this maximum to lie at W ~ 0.07 MeV (which appears to be twice that indicated by Evans, i.e. 37 keV). The cross sections agree well with Evans' graph, thus:-

photon energy (B+W), MeV	**total CoM product nucleon energy (W), MeV**	σ_{dis}^{E} **(millibarn)**
2.3	0.076	0.11
2.5	0.276	0.58
3	0.776	1.57
4	1.776	2.30
4.44	**2.216**	**2.34**
6	3.776	2.11
8	5.776	1.68
14	11.776	0.91

(maximum cross-section shown in bold)

photon energy (B+W), MeV	**total CoM product nucleon energy (W), MeV**	σ_{dis}^{M} **(millibarn)**
2.229	0.005	0.28
2.234	0.01	0.37
2.244	0.02	0.47
2.254	0.03	0.52
2.264	0.04	0.54
2.274	0.05	0.550
2.294	**0.07**	**0.558**
2.314	0.09	0.551
2.334	0.11	0.538
2.354	0.13	0.523
2.424	0.2	0.47
3	0.776	0.24
6	3.776	0.06

(maximum cross-section shown in bold)

Having established that this method of calculation gives reliable results for the $p + n \Leftrightarrow D + \gamma$ cross-sections in this universe, we are confident to go forward in Appendix A4 to consider how the same approach may be modified to give the cross sections for all two-nucleon reactions in a universe with a stronger nuclear force. Specifically we shall be interested in the cross sections for $p + p \Leftrightarrow {}_{2}^{2}He + \gamma$. Before looking at the diproton production rate, however, we shall first adapt the method to estimate the rate of the reaction $p + D \rightarrow {}_{2}^{3}He + \gamma$ in Appendix A3.