Appendix A1 – Dependence of Neutron Lifetime on Weak Interaction Strength

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1. Introduction

The main purpose of this Appendix is to show that the lifetime of a free neutron, decaying through beta decay, is inversely proportional to the square of the weak interaction coupling strength, G_F. We also show how the neutron lifetime may be estimated, at least approximately. Three approximate calculations are presented, in order of increasing accuracy.

The first estimate, in Section 3, is based on a naïve analogy with lepton decay, which is reviewed in Section 2. This suffers from a lack of clarity regarding the effective mass to employ (properly, the relevant phase space factor).

The second estimate, in Section 4, is an exact calculation of the first order Feynman decay rate based on a pure V-A hadronic weak current and point-like nucleons with no structure. This is a genuine calculation, albeit based on grossly oversimplified assumptions. It over-estimates the neutron lifetime by ~50%.

Finally a crude attempt is made in Section 5 to account for the hadronic structure of the nucleons. This is done extremely simply by appeal to the Goldberger-Treiman relation. The result is a remarkably good estimate of the neutron mean life, in error by less than 1%. Since Goldberger-Treiman is based on a one-pion exchange correction to the $\langle n|pe\nu\rangle$ vertex, which is hardly a precise account of the hadronic structure, this close agreement is probably more a lucky fluke than a genuinely accurate estimate. In truth an electromagnetic (radiative) correction should really be included, which other sources quote as amounting to several percent. Since this has been ignored here we should really only expect an accuracy of a few percent.

2. Lepton Decay

The formula for muon decay, derived in Commins, Equ.(2.41), or Mandl and Shaw, Equ.(11.69) is,

$$\tau_\mu = \frac{192\pi^3}{G_F^2 m_\mu^5} \tag{1}$$

where G_F is the Fermi constant, $G_F \approx 10^{-5}/M_p^2$. The mass of the muon appears in the denominator of (1) raised to the power 5. The reason for the power of 5 can be seen from dimensional analysis. The Feynman amplitude for the decay, in terms of the simple Fermi point-interaction Lagrangian, is just a single vertex and hence is proportional to G_F. Hence, the transition rate, dependent on the square of the amplitude, is proportional to G_F^2. Thus, the lifetime is proportional to $1/G_F^2$.

Now, the interaction Lagrangian, with units of energy per volume, equals G_F times a product of four fermion fields. The square of a fermion field has units "per unit volume", thus G_F must have units "energy x volume". In the convention $\hbar = c = 1$, for which L = T = 1/E = 1/M, G_F thus has units $1/E^2$ or $1/M^2$, according to taste. This is consistent with $G_F \approx 10^{-5}/M_p^2$. Now, in Equ.(1) we need to achieve units of T $\equiv 1/E$. Thus, since $1/G_F^2$ has units E^4, we need to divide by E^5, hence the power 5 dependence on the mass. Actually, there is another mass in the problem, namely the electron mass. The derivation of (1) involves neglecting the electron mass

(0.511MeV) compared with the muon mass (105.7MeV). In general, any function of the two masses with dimension M^5 could be permitted by dimensional analysis. (Field theory can, however, be used to derive specifically what function – and hence to show that Equ.1 is correct in the approximation that $m_e << m_\mu$).

Incidentally, this derivation by dimensional analysis provides a counter example to a familiar prejudice. It is often claimed (see for example Barrow & Tipler P.270, 290) that the numerical coefficients arising in dimensional analysis tend to be of order unity. Without this assumption, dimensional analysis alone cannot be used to estimate even the order of magnitude of a quantity. Now Equ.(1) is clearly a counter example since the numerical coefficient is nearly 4 orders of magnitude greater than unity. The moral is that dimensional analysis must be treated with great caution. It can mislead by four orders of magnitude – and perhaps far worse.

Nevertheless, dimensional analysis is still very useful even in this case. For example it implies that the lifetime of the tau meson, of mass 1777 MeV, cannot be greater than $(105.7 / 1777)^5 = 7.4 \times 10^{-7}$ times the lifetime of the muon. Since the muon lifetime is 2.2 microsec, this implies that the tau lifetime cannot be greater than 1.6×10^{-12} sec. This is correct. Actually, we can immediately conclude that the tau lifetime cannot be greater than half this value, since in addition to the beta decay involving the production of an electron, the tau can equally decay into a muon, and the transitions rates for the two will be of the same. Actually, the branching ratio of the tau decay into each of these channels is ~0.176 +/- 0.002. Given this we would predict the tau lifetime to be 1.6×10^{-12} sec x 0.176 = 2.8×10^{-13} sec, which is in good agreement with the experimental value (2.9×10^{-13} sec).

3. Argument For Neutron Decay By Analogy With Lepton Decay

So much for lepton decays. What about neutron decay? Well, the dimensional argument above clearly establishes what we wanted to know most, namely that the neutron lifetime must also be proportional to $1/G_F^2$. Can we do better and calculate it? The simplest guess is that the lepton lifetime formula holds, except with the mass in the denominator replaced by…..what? Clearly it would not be appropriate to simply insert the neutron mass, since beta decay of the neutron creates an almost-as-massive proton. Recall that the derivation of Equ.1 involves neglecting the mass of the product particle. This is clearly inappropriate in this case. The decay will obviously be phase-space suppressed if the mass difference is sufficiently small, since, in the limit of zero mass difference there can be no decay. Hence, we guess that replacing the mass in the denominator of (1) by the neutron/proton mass difference, $M_n – M_p = 1.2934$ MeV, may give an order of magnitude estimate.

$$\tau_n \approx \frac{192\pi^3}{G_F^2(1.2934\,\mathrm{MeV})^5} = 1.27\mathrm{x}10^{25}\ \mathrm{MeV}^{-1} = 8400\,\mathrm{sec.} \qquad (2)$$

where we have used $G_F \approx 10^{-5} / \left(M_p = 938.3\,\mathrm{MeV}\right)^2$ and the conversion factor

1 $\mathrm{MeV}^{-1} = 6.59 \times 10^{-22}$ sec. This is an order of magnitude too long, the actual neutron lifetime being 886 sec. In fact things are even worse than this. The correct "phase space" factor must depend upon $M_n – M_p – m_e = 0.7824$ MeV, not $M_n – M_p = 1.2934$ MeV, in which case the estimated lifetime would be larger still.

However, we note that neutron beta decay is best envisaged as a decay of a 'down' quark into an 'up' quark. This suggests two things: firstly that the relevant phase space factor would involve the quark masses, specifically $M_{down} - M_{up}$ - m_e. Secondly, because there are two down quarks per neutron the decay rate per neutron will be doubled, i.e. the lifetime halved. Unfortunately, the masses of the quarks are not well known, the ranges given in the 2010 Particle Data Group summary tables being M_{down} = 4.1 to 5.8 MeV, and M_{up} = 1.7 to 3.3 MeV. If the bounds were perfectly correlated the quark mass difference would lie in the range 2.4 to 2.5 MeV. This gives $M_{down} - M_{up}$ - m_e ~ 1.94 MeV and hence,

$$\tau_n \approx \frac{1}{2}\frac{192\pi^3}{G_F^2 (1.94\, MeV)^5} = 8.4\times 10^{23}\ \mathrm{MeV}^{-1} = 553\,\mathrm{sec.} \tag{3}$$

This is not a bad estimate. But we can hardly claim any great success because the quark mass difference could be anywhere between 0.8 MeV and 4.1 MeV, giving a range of possible neutron lifetimes of 13 seconds to 46,000 seconds

If an equation like (3) were indeed accurate it would provide a rather accurate means of finding the down/up quark mass difference. Using the slightly more accurate $G_F = \frac{1.026\times 10^{-5}}{M_p^2} = 1.166\times 10^{-11} MeV^{-2}$ we need $M_{down} - M_{up}$ - m_e = 1.747 MeV to reproduce the neutron lifetime of 886 seconds. Hence $M_{down} - M_{up}$ = 1.236 MeV. Unfortunately Equ.3 is probably far too simplistic. Nevertheless, we have achieved our aim in rationalising the lifetime of the neutron approximately in terms of the strength of the weak interaction, in particular $\tau_n \propto \frac{1}{G_F^2}$.

4. Mean Life of a Structureless Neutron with a V-A Weak Current

This next estimate of the neutron mean life is based on the fiction that the nucleons are point-like and without structure, and that the hadronic weak current is simply $\overline{\psi}_p \gamma_\alpha \left(1+\gamma^5\right)\psi_n$. With the usual definition of the Fermi constant the matrix element is,

$$\begin{aligned} |M|^2 &= \frac{G_F^2}{2}\left[\overline{u}_e \gamma_\alpha \left(1+\gamma^5\right) v \overline{v} \gamma_\beta \left(1+\gamma^5\right) u_e\right]\left[\overline{u}_p \gamma^\alpha \left(1+\gamma^5\right) u_n \overline{u}_n \gamma^\beta \left(1+\gamma^5\right) u_p\right] \\ &= 16 G_F^2 \left[\left(p_e - m_e s_e\right)\cdot\left(p_p + M_p s_p\right)\right]\left[q\cdot\left(p_n + M_n s_n\right)\right] \end{aligned} \tag{4}$$

The latter results from the former using the usual trace formulae, and expressions like $u_e \overline{u}_e = \frac{1}{2}\left(\not{p}_e + m_e\right)\left(1-\gamma^5 \not{s}_e\right)$, etc. , where the sign convention is $\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$. The Annex at the end of this Appendix shows in detail how this can be derived.

The differential decay rate is: $$dW = \frac{\delta^4\left(p_p + p_e + q - p_n\right)}{16(2\pi)^5 E_n E_p E_e E_\nu}|M|^2 d^3 p_p d^3 p_e d^3 q \tag{5}$$

Carrying out the p_p integral in the neutron rest frame gives,

$$dW = \frac{\delta\left(E_p + E_e + E_\nu - M_n\right)}{(2\pi)^5 M_n E_p E_e E_\nu} G_F^2 \left[\left(p_e - m_e s_e\right)\cdot\left(p_p - M_p s_p\right)\right]\left[q\cdot\left(p_n - M_n s_n\right)\right] d^3 p_e d^3 q \tag{6}$$

where it is now understood that $\overline{p}_p = -\left(\overline{p}_e + \overline{q}\right)$.

We wish to average over the neutron spins and to sum over the proton and electron spins, so we want to replace dW with $\frac{1}{2}\sum_{s_n,s_p,s_e} dW$. Hence,

$$dW_{av} = \frac{4G_F^2\delta\left(E_p + E_e + E_\nu - M_n\right)}{(2\pi)^5 M_n E_p E_e E_\nu}\left[(p_e)\cdot(p_p)\right]\left[q\cdot(p_n)\right]d^3p_e d^3q \qquad (7)$$

A sufficient approximation is $E_p \approx M_p$, and hence,

$$q\cdot p_n = q\cdot\left(M_n,\bar{0}\right) = M_n E_\nu \qquad (8)$$

and

$$p_e\cdot p_p = p_e\cdot\left(M_p, -(\bar{p}_e + \bar{q})\right) = E_e M_p + \bar{p}_e\cdot(\bar{p}_e + \bar{q}) \qquad (9)$$

Hence,

$$dW_{av} = \frac{4G_F^2\delta\left(E_e + E_\nu - \Delta\right)}{(2\pi)^5 M_p E_e}\left[E_e M_p + \bar{p}_e\cdot(\bar{p}_e + \bar{q})\right]d^3p_e d^3q \qquad (10)$$

The term in $\bar{p}_e\cdot\bar{q}$ integrates over q to zero. The other integral over q which is required is simply $\int\delta\left(E_e + E_\nu - \Delta\right)d^3q = 4\pi\int\delta\left(E_e + E_\nu - \Delta\right)E_\nu^2 dE_\nu = 4\pi(\Delta - E_e)^2$, where $\Delta = M_n - M_p$. Hence,

$$dW_{av} = \frac{G_F^2}{2\pi^4}\left[1 + \frac{|\bar{p}_e|^2}{M_p E_e}\right](\Delta - E_e)^2 d^3p_e \qquad (11)$$

The second term in [...] can be neglected, so,

$$W = \frac{G_F^2}{2\pi^4}\int(\Delta - E_e)^2 d^3p_e = \frac{2G_F^2}{\pi^3}\int(\Delta - E_e)^2 p_e^2 dp_e = \frac{2G_F^2}{\pi^3}\int_{m_e}^{\Delta}(\Delta - E_e)^2 p_e E_e dE_e \qquad (12)$$

The last expression shows the limits of the integral expressed as an integral over electron energy, i.e., the limits are clearly $[m_e, \Delta]$. Hence, the neutron mean life is,

$$\tau = \frac{1}{W} = \frac{\pi^3}{2fG_F^2} \qquad \text{where,} \qquad f = \int_{m_e}^{\Delta}(\Delta - E_e)^2 p_e E_e dE_e \qquad (13)$$

The integral, *f*, is easily evaluated in the non-relativistic approximation, giving,

$$f \approx \sqrt{2}m_e^{3/2}\int_{m_e}^{\Delta}(\Delta - E_e)^2\sqrt{E_e - m_e}\cdot dE_e = \frac{16\sqrt{2}}{105}m_e^{3/2}\tilde{\Delta}^{7/2} = 0.0333MeV^5 \qquad (14)$$

where $\tilde{\Delta} = \Delta - m_e = 0.7823MeV$. This is the correct phase space factor in the non-relativistic approximation. However, this is a poor approximation: the electron is actually likely to be relativistic because $\Delta > m_e$.

The *f* integral can be evaluated numerically using the relativistic momentum. This gives the accurate value to be $f = 0.057MeV^5$ (using $\Delta = 1.2933MeV$, $m_e = 0.511MeV$).

Using $G_F = 1.166\times10^{-11}MeV^{-2}$ thus gives an estimated neutron lifetime of,

2.0×10^{24} MeV^{-1} = **1318 seconds** cf. the actual mean life of 886 seconds. Not too bad really, considering the crude model used. However, it is possible to do much better without undue effort…

5. Hadronic Structure Correction to Neutron Lifetime Estimate

- The previous estimate was based on the structureless V-A hadronic weak current $\overline{\psi}_p \gamma_\alpha \left(1+\gamma^5\right)\psi_n$. This produces a neutron mean life of $\tau = \frac{1}{W} = \frac{\pi^3}{2 f G_F^2}$ of 1318 seconds (where $f = 0.057 MeV^5$).
- The most general hadronic weak current would be a linear combination of six terms: $\gamma_\alpha, q_\alpha, \sigma_{\alpha\beta} q^\beta, \gamma_\alpha \gamma_5, q_\alpha \gamma_5$ and $\sigma_{\alpha\beta} q^\beta \gamma_5$. In general the coefficients of these terms would be six independent functions of the scalar of 4-momentum transfer, q^2. However, in beta decay q^2 is modest and these functions may be treated as constants, i.e., equal to their values at $q^2 = 0$. Various arguments might be used to try to constrain the values of these constants. However, the comparatively small momentum transfer, q (compared to the nucleon mass) is sufficient to show that the contribution of $q_\alpha, \sigma_{\alpha\beta} q^\beta, q_\alpha \gamma_5$ and $\sigma_{\alpha\beta} q^\beta \gamma_5$ are negligible.
- Consequently, the effectively hadronic weak current reduces to $\overline{\psi}_p \gamma_\alpha \left(C_V + C_A \gamma^5\right)\psi_n$.
- Empirically it appears that $C_V \approx 1$. This conclusion is also reached by assuming that the vector current is conserved, in analogy to that of electric current. Strictly this is not obvious, but it provides a reasonable rationalisation of this assumption. The experimental evidence essentially suggests that QCD results in no renormalisation of this coupling, which is just a clever way of saying that it remains $C_V = 1$ despite the hadronic structure.
- However, C_A is not unity (the axial current is not conserved, but only partially conserved). Fortunately there is an existing theory for C_A, namely the Goldberger-Treiman relation. This is based on a one-pion exchange correction to the $\langle n | pe\upsilon \rangle$ vertex and gives, $C_A = \frac{\sqrt{2} f_\pi g_0}{M_p + M_n}$, where: f_π is the weak vertex form factor for the charged pion, whose lifetime gives $f_\pi = 0.93 m_\pi$, and g_0 is the low energy strong nuclear coupling defined via the pion-nucleon vertex.
- For $g_0^2 / 4\pi$ we shall use the average of four values given recently, as follows,
  - 13.5 (de Swart, 1998, nucl-th/9802084)
  - 14.1 (Ericson, 2002, Phys. Rev. C 66, 014005 [2002])
  - 13.8 (Bugg, 2004, Eur.Phys.J. C 33, 505-509 [2004])
  - 13.7 (Pavan, 1999, nucl-th/9910040)

giving an average $g_0^2 / 4\pi$ of 13.8, and hence $g_0 = 13.2$.

- Hence Goldberger-Treiman gives $C_A = 1.29$. This compares with the value of 1.267 quoted by Byrne[1], Equ.(47), so is not too bad.
- Such slightly different values for C_A can be found elsewhere in the literature. But these are generally derived using beta decay data, or inverse beta decay data, of the neutron or of nuclei. But since it is a beta decay process which we trying to predict, to use beta decay data would involve circular reasoning. The advantage of the above estimate of C_A (1.29) is that it does not use any form of beta decay data, being based only on the pion lifetime and the pion-nucleon coupling (and noting that G_F itself may be taken from muon decay). Hence there is no circular logic involved.
- Working through the derivation of the neutron decay rate using $\bar{\psi}_p \gamma_\alpha \left(1 + C_A \gamma^5\right)\psi_n$ gives the same result as before but multiplied by $\left(1 + 3|C_A|^2\right)/4$. The details of the spinor algebra which leads to this result are given in the Annex below. Consequently the decay rate increases by a factor $\left(1 + 3 \times 1.29^2\right)/4 = 1.50$
- The estimate of neutron lifetime accounting for hadronic structure via one pion exchange (i.e., Goldberger-Treiman) is thus $\tau = \frac{1}{W} = \frac{\pi^3}{3 f G_F^2} =$ **880 seconds**, or virtually spot-on the actual 886 seconds mean life.
- This is amazingly good – too good in fact. In truth there is a large element of dumb luck in getting so close to the right answer by this route. For one thing, the Goldberger-Treiman relation is only approximate. A one pion exchange model of hadron structure is hardly going to be very accurate! (Although it is worth noting that, if the relation itself were accurate, the error in C_A due to uncertainty in $f_\pi g_0$ is probably only around 1.7%, or ~15 seconds on the neutron life). However, the neutron decay rate should be corrected for radiation effects. Commins[2] states that the total correction can amount to several percent, and this is confirmed by Byrne's Equs.(24-26) which imply a correction of ~5%. So, if we did not already know the answer, we would only be entitled to conclude that the neutron lifetime is expected to be within a few percent of 880 seconds.

Annex – Details of the Derivation of the Final Decay Rate Formula

Putting $\lambda = C_A$ for brevity, the matrix element becomes,

$$|M|^2 = \frac{G_F^2}{2}\left[\bar{u}_e \gamma_\alpha \left(1+\gamma^5\right) v \bar{v} \gamma_\beta \left(1+\gamma^5\right) u_e\right]\left[\bar{u}_p \gamma^\alpha \left(1+\lambda\gamma^5\right) u_n \bar{u}_n \gamma^\beta \left(1+\lambda\gamma^5\right) u_p\right] \qquad \text{(A1)}$$

The first (leptonic) term is unchanged. It can be evaluated using the trace formulae for the gamma matrices as follows.

$$[1] = \left[\bar{u}_e \gamma_\alpha \left(1+\gamma^5\right) v \bar{v} \gamma_\beta \left(1+\gamma^5\right) u_e\right] = Tr\left[u_e \bar{u}_e \gamma_\alpha \left(1+\gamma^5\right) v \bar{v} \gamma_\beta \left(1+\gamma^5\right)\right] \qquad \text{(A2)}$$

Now we use the result that,

$$u_e \bar{u}_e = \frac{1}{2}\left(\not{p}_e + m_e\right)\left(1 - \gamma^5 \not{s}_e\right) \qquad \text{(A3)}$$

where the sign convention is $\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$. Expressions like (A3) are, of course, also true for the proton and the neutron, whilst, for the neutrinos it is

[1] www.physi.uni-heidelberg.de/Publications/ckm_byrne.pdf
[2] Eugene D Commins, "Weak Interactions", McGraw-Hill, 1973.

$$u_\nu \bar{u}_\nu = \not{q} \equiv q_\alpha \gamma^\alpha \tag{A4}$$

So Equ.(2) becomes,

$$\begin{aligned}
[1] &= Tr\left[\frac{1}{2}\left(\not{p}_e + m_e\right)\left(1-\gamma^5 \not{s}_e\right)\gamma_\alpha\left(1+\gamma^5\right)\not{q}\gamma_\beta\left(1+\gamma^5\right)\right] \\
&= Tr\left[\left(\not{p}_e + m_e\right)\left(1-\gamma^5 \not{s}_e\right)\gamma_\alpha \not{q}\gamma_\beta\left(1+\gamma^5\right)\right] \\
&= Tr\left[\left(\not{p}_e - m_e\gamma^5 \not{s}_e\right)\gamma_\alpha \not{q}\gamma_\beta\left(1+\gamma^5\right)\right] \\
&= Tr\left[\left(\not{p}_e - m_e \not{s}_e\right)\gamma_\alpha \not{q}\gamma_\beta\left(1+\gamma^5\right)\right] \\
&= Tr\left[\left(p_e^\mu - m_e s_e^\mu\right)\gamma_\mu\gamma_\alpha q^\nu \gamma_\nu \gamma_\beta\left(1+\gamma^5\right)\right] \\
&= 4\left(p_e^\mu - m_e s_e^\mu\right)q^\nu \chi_{\mu\nu\alpha\beta}
\end{aligned} \tag{A5}$$

(A5) has been evaluated using $\left(\gamma^5\right)^2 = 1$, and hence $\gamma^5\left(1+\gamma^5\right) = \left(1+\gamma^5\right)$ and $\left(1+\gamma^5\right)^2 = 2\left(1+\gamma^5\right)$, together with the fact that the trace of an odd number of γ^α is zero, with or without an additional factor of γ^5, and also $\gamma^5\gamma^\alpha = -\gamma^\alpha\gamma^5$. Finally, the non-zero trace is given by,

$$\chi_{\mu\nu\alpha\beta} \equiv \frac{1}{4} Tr\left[\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta\left(1+\gamma^5\right)\right] = g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha} - g_{\mu\nu}g_{\alpha\beta} - i\varepsilon_{\mu\alpha\nu\beta} \tag{A6}$$

This and the other trace formulae will not be proved here but can be found in most texts on field theory, e.g., Commins or Bjorken & Drell. When summed over the two electron polarisations, (A5) becomes,

$$[1] = 8 p_e^\mu q^\nu \chi_{\mu\nu\alpha\beta} \tag{A7}$$

The second term in Equ.(A1) is,

$$[2] \equiv \left[\bar{u}_p \gamma^\alpha\left(1+\lambda\gamma^5\right)u_n \bar{u}_n \gamma^\beta\left(1+\lambda\gamma^5\right)u_p\right] = Tr\left[u_p \bar{u}_p \gamma^\alpha\left(1+\lambda\gamma^5\right)u_n \bar{u}_n \gamma^\beta\left(1+\lambda\gamma^5\right)\right] \tag{A8}$$

Hence,

$$\begin{aligned}
[2] &= Tr\left[\frac{1}{2}\left(\not{p}_p + M_p\right)\left(1-\gamma^5 \not{s}_p\right)\gamma^\alpha\left(1+\lambda\gamma^5\right)\frac{1}{2}\left(\not{p}_n + M_n\right)\left(1-\gamma^5 \not{s}_n\right)\gamma^\beta\left(1+\lambda\gamma^5\right)\right] \\
&= \frac{1}{4} Tr\left[\begin{array}{l}\left(\not{p}_p - M_p\gamma^5 \not{s}_p\right)\gamma^\alpha\left(1+\lambda\gamma^5\right)\left(\not{p}_n - M_n\gamma^5 \not{s}_n\right)\gamma^\beta\left(1+\lambda\gamma^5\right) + \\ \left(M_p + \gamma^5 \not{p}_p \not{s}_p\right)\gamma^\alpha\left(1+\lambda\gamma^5\right)\left(M_n + \gamma^5 \not{p}_n \not{s}_n\right)\gamma^\beta\left(1+\lambda\gamma^5\right)\end{array}\right]
\end{aligned} \tag{A9}$$

When summed over the two proton polarisations and averaged over the two neutron polarisations, (A9) becomes,

$$\begin{aligned}
[2] &= \frac{1}{2} Tr\left[p_{p\gamma} p_{n\delta} \gamma^\gamma \gamma^\alpha\left(1+\lambda\gamma^5\right)\gamma^\delta\gamma^\beta\left(1+\lambda\gamma^5\right) + M_p\gamma^\alpha\left(1+\lambda\gamma^5\right)M_n\gamma^\beta\left(1+\lambda\gamma^5\right)\right] \\
&= \frac{1}{2} Tr\left[p_{p\gamma} p_{n\delta} \gamma^\gamma \gamma^\alpha \gamma^\delta \gamma^\beta\left(1+\lambda^2+2\lambda\gamma^5\right) + M_p M_n \gamma^\alpha\gamma^\beta\left(1-\lambda^2\right)\right]
\end{aligned} \tag{A10}$$

Now we use the trace formulae,

$$\frac{1}{4} Tr\left[\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta\right] = g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha} - g_{\mu\nu}g_{\alpha\beta} \tag{A11}$$

$$\frac{1}{4}Tr\left[\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta\gamma^5\right] = -i\varepsilon_{\mu\alpha\nu\beta} \qquad \text{(A12)}$$

$$\frac{1}{4}Tr\left[\gamma_\mu\gamma_\alpha\right] = g_{\mu\alpha} \qquad \text{(A13)}$$

So that (A10) becomes, (A14)

$$[2] = 2\left[\left(1+\lambda^2\right)p_{p\gamma}p_{n\delta}\left(g^{\gamma\alpha}g^{\delta\beta} + g^{\gamma\beta}g^{\delta\alpha} - g^{\gamma\delta}g^{\alpha\beta}\right) - 2i\lambda p_{p\gamma}p_{n\delta}\varepsilon^{\gamma\alpha\delta\beta} + \left(1-\lambda^2\right)M_pM_ng^{\alpha\beta}\right]$$

From (A1) we have $|M|^2 = \frac{G_F^2}{2}[1][2]$ where the two terms are given by (A7) and (A14), after suitable spin summing/averaging. The contractions in this expression can be evaluated using,

$$\chi_{\mu\nu\alpha\beta}g^{\alpha\beta} = -2g_{\mu\nu} \qquad \text{(A15)}$$

$$\chi_{\mu\nu\alpha\beta}\left(g^{\gamma\alpha}g^{\delta\beta} + g^{\gamma\beta}g^{\delta\alpha} - g^{\gamma\delta}g^{\alpha\beta}\right) = 2\left(g_\mu^\gamma g_\nu^\delta + g_\mu^\delta g_\nu^\gamma\right) \qquad \text{(A16)}$$

$$\chi_{\mu\nu\alpha\beta}\left(-i\varepsilon^{\gamma\alpha\delta\beta}\right) = 2\left(g_\mu^\gamma g_\nu^\delta - g_\mu^\delta g_\nu^\gamma\right) \qquad \text{(A17)}$$

Hence we get,

$$\frac{1}{2}\sum_{s_n,s_p,s_e}|M|^2 = \frac{G_F^2}{2}\times 8p_e^\mu q^\nu \times 2\times 2\begin{bmatrix}\left(1+\lambda^2\right)p_{p\gamma}p_{n\delta}\left(g_\mu^\gamma g_\nu^\delta + g_\mu^\delta g_\nu^\gamma\right) + \\ 2\lambda p_{p\gamma}p_{n\delta}\left(g_\mu^\gamma g_\nu^\delta - g_\mu^\delta g_\nu^\gamma\right) - \left(1-\lambda^2\right)M_pM_ng_{\mu\nu}\end{bmatrix} \qquad \text{(A18)}$$

Hence,

$$\frac{1}{2}\sum_{s_n,s_p,s_e}|M|^2 = 16G_F^2\begin{bmatrix}\left(1+\lambda^2\right)\left\{\left(p_e\cdot p_p\right)\left(q\cdot p_n\right) + \left(p_e\cdot p_n\right)\left(q\cdot p_p\right)\right\} + \\ 2\lambda\left\{\left(p_e\cdot p_p\right)\left(q\cdot p_n\right) - \left(p_e\cdot p_n\right)\left(q\cdot p_p\right)\right\} - \left(1-\lambda^2\right)M_pM_nq\cdot p_e\end{bmatrix} \qquad \text{(A19)}$$

Equ.(A19) is exact. We now introduce approximations, (A20)

$$\begin{aligned}\left(p_e\cdot p_p\right)\left(q\cdot p_n\right) + \left(p_e\cdot p_n\right)\left(q\cdot p_p\right) &= M_nE_\nu\left(E_eE_p + \bar{p}_e\cdot\left(\bar{q}+\bar{p}_e\right)\right) + M_nE_e\left(E_\nu E_p + \bar{q}\cdot\left(\bar{q}+\bar{p}_e\right)\right) \\ &\approx 2M_nE_pE_\nu E_e\end{aligned}$$

(A21)

$$\begin{aligned}\left(p_e\cdot p_p\right)\left(q\cdot p_n\right) - \left(p_e\cdot p_n\right)\left(q\cdot p_p\right) &= M_nE_\nu\left(E_eE_p + \bar{p}_e\cdot\left(\bar{q}+\bar{p}_e\right)\right) - M_nE_e\left(E_\nu E_p + \bar{q}\cdot\left(\bar{q}+\bar{p}_e\right)\right) \\ &\approx 0\end{aligned}$$

$$q\cdot p_e = E_\nu E_e - \bar{q}\cdot\bar{p}_e \qquad \text{(A22)}$$

Equ.(A20) follows from the fact that p_e and q cannot exceed $\Delta = M_n - M_p = 1.2933$ MeV, whereas M_n and $E_p \approx M_p$ are about 700 times larger. Hence the approximation is probably accurate to ~0.1% or so. For the same reason, Equ.(A21) can be replaced by zero, being 3 orders of magnitude smaller than Equ.(A20).

Equ.(A22) is exact. However, when the matrix element is integrated over the 3-momenta, $\bar{q}$ and $\bar{p}_e$, to form the total decay rate, the dot product term becomes zero. With this in mind we can therefore write (A19) as,

$$\begin{aligned}\frac{1}{2}\sum_{s_n,s_p,s_e}|M|^2 &= 16G_F^2\left[\left(1+\lambda^2\right)\left\{2M_nE_pE_\nu E_e\right\}-\left(1-\lambda^2\right)M_pM_nE_\nu E_e\right]\\ &\approx 16G_F^2\left[2\left(1+\lambda^2\right)-\left(1-\lambda^2\right)\right]M_nM_pE_\nu E_e \\ &= 16G_F^2\left(1+3\lambda^2\right)M_nM_pE_\nu E_e\end{aligned} \quad \text{(A23)}$$

Note that because we have already done the spin summing/averaging: $\frac{1}{2}\sum_{s_n,s_p,s_e} dW$, this result is actually to be compared with 4x the point-like hadron result for $|M|^2$, i.e., 4x Equ.(4), or $4|M|^2 = 64G_F^2\left(p_e\cdot p_p\right)\left(q\cdot p_n\right)\approx 64G_F^2M_nM_pE_\nu E_e$. So we see that the square-matrix element, and hence the decay rate, when a non-unity axial form factor is included in the hadronic weak current, is a factor $\left(1+3\lambda^2\right)/4$ greater than estimated in Section 4. The final expression for the neutron lifetime is thus,

$$\tau = \frac{1}{W} = \frac{2\pi^3}{f\left(1+3\lambda^2\right)G_F^2} \quad \text{(A42a)}$$

where, $$\lambda = \frac{\sqrt{2}f_\pi g_0}{M_p+M_n} = 1.29 \text{ (Goldberger-Treiman)} \quad \text{(A42b)}$$

and, $$f_\pi g_0 = 0.93\times 13.2 \quad \text{(from pion physics)} \quad \text{(A24c)}$$

and, $$f = \int_{m_e}^{\Delta}\left(\Delta-E_e\right)^2 p_eE_edE_e = 0.0570MeV^5 \text{ (relativistic)} \quad \text{(A24d)}$$

and, $$\Delta = M_n - M_p = 1.2933 \text{ MeV.} \quad \text{(A24e)}$$

and, $$G_F = 1.166\times 10^{-11}MeV^{-2} \quad \text{(e.g., from muon decay)} \quad \text{(A24f)}$$

and, $$1MeV^{-1} = 6.463\times 10^{-22} \text{ seconds} \quad \text{(A24g)}$$

Hence, $\tau =$ **880 seconds**, cf., the measured mean life of 886 seconds.